Name: (in BLOCK CAPITALS) ________________________________

Signature: ____________________________________________

Instructions

This is a closed-book closed-notes examination except that one side of one 8.5” × 11” sheet of notes is permitted. Tables of integrals, calculators, computers, telephones, microscopes, etc. are neither needed nor permitted.

SHOW YOUR WORK. If you need additional space, use the back of the previous page. Write your final answers in the spaces provided.

This exam contains four problems on seven separate pages

<table>
<thead>
<tr>
<th>Grading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 24 points ____________</td>
</tr>
<tr>
<td>2. 24 points ____________</td>
</tr>
<tr>
<td>3. 28 points ____________</td>
</tr>
<tr>
<td>4. 24 points ____________</td>
</tr>
<tr>
<td>Total (100 points) ____________</td>
</tr>
</tbody>
</table>
1. Consider a binary digital communication system with intersymbol interference. Using our standard discrete-time model, the channel output is:

\[ y[m] = x[m] + x[m-1] + w[m], \quad m \geq 1, \]

where \( x[m] \in \pm \sqrt{E} \) (which are chosen equiprobably) and \( w[m] \) is an i.i.d. noise sequence \( \sim \mathcal{N}(0, \sigma^2) \).

We use successive interference cancellation at the receiver.

(a) Suppose the first two transmitted signals are \( x[1] = \sqrt{E} \) and \( x[2] = \sqrt{E} \), and we are able to decode them correctly as \( \hat{x}[1] = \sqrt{E} \) and \( \hat{x}[2] = \sqrt{E} \). Design a computation for \( \hat{x}[3] \) (the decoded version of \( x[3] \)), which minimizes the probability of error. The receiver has access to \( \hat{x}[1], \hat{x}[2], y[1], y[2], y[3] \) and may use some or all of them.

**Solution**

Let us use \( y[3] \) and \( \hat{x}[2] \) to design our decoder. In particular, let us find:

\[
\]

where the last step follows since \( \hat{x}[2] = x[2] \). Now we are just facing a standard AWGN channel, and by symmetry, the optimal decision rule is based on the sign:

\[
\hat{x}[3] = \begin{cases} 
\sqrt{3}, & y[3] - \hat{x}[2] > 0 \\
-\sqrt{3}, & y[3] - \hat{x}[2] \leq 0.
\end{cases}
\]

(b) What is the average error probability, \( P_e \), achieved by your computation in part (a)?

\[
P_e = Q(\sqrt{E}/\sigma)
\]

**Solution**

This just reduces to a standard AWGN problem, with signals \( \pm \sqrt{E} \) and noise variance \( \sigma^2 \), so \( P_e = Q(\sqrt{E}/\sigma) \).
(c) Suppose the first two transmitted signals are \(x[1] = \sqrt{E}\) and \(x[2] = \sqrt{E}\), and we decode them both incorrectly as \(\hat{x}[1] = -\sqrt{E}\) and \(\hat{x}[2] = -\sqrt{E}\). (Of course the receiver does not know that the decoding was incorrect.) Now what is the average error probability, \(P_e\), achieved by your computation in part (a)?

\[
P_e = \frac{1}{2} Q(\frac{3\sqrt{E}}{\sigma}) + \frac{1}{2} Q(-\frac{\sqrt{E}}{\sigma})
\]

Solution

We use the definition of \(\tilde{y}[3]\) and decoding rule from before:

\[
\hat{x}[3] = \begin{cases} 
\sqrt{3}, & \tilde{y}[3] > 0 \\
-\sqrt{3}, & \tilde{y}[3] \leq 0.
\end{cases}
\]

Due to the error, rather than cancelling interference, we are in a sense biasing the system. We will have \(\tilde{y}[3] = x[3] + w[3] + 2\sqrt{3}\). Consider the two possibilities for \(x[3]\) separately.

If \(x[3] = \sqrt{E}\), then \(\tilde{y}[3] = 3\sqrt{E} + w[3]\), and there is an error event if \(w[3] \leq -3\sqrt{E}\). That is to say \(Q(3\sqrt{E}/\sigma)\). If \(x[3] = -\sqrt{E}\), then \(\tilde{y}[3] = \sqrt{E} + w[3]\), and there is an error event if \(w[3] > -\sqrt{E}\). That is to say \(Q(-\sqrt{E}/\sigma)\). Since each transmission is equiprobable, the error probability is \(\frac{1}{2} Q(3\sqrt{E}/\sigma) + \frac{1}{2} Q(-\sqrt{E}/\sigma)\).

(d) Suppose there is an i.i.d. sequence of Bernoulli random variables with probability of one being \(Q(4)\) and probability of zero being \(1 - Q(4)\). What is the probability, \(p\), of getting 49 ones in a row, and then a zero?

\[
p = q^{49}(1 - q)
\]

Solution

This is just a binomial random variable. Let \(q = Q(4)\), then \(p = q^{49}(1 - q)\).
2. The Parsons Code for Melodic Contours is a simple notation used to identify a piece of music through melodic motion—the motion of the pitch up and down. Representing a melody in this manner makes it easy to index or search for particular pieces, and is used e.g. as the internal representation in matched-filter software such as Shazam.

Notes are denoted with one of three letters to indicate the relationship of its pitch to the previous note:

- U = “up,” if the note is higher than the previous note,
- D = “down,” if the note is lower than the previous note,
- R = “repeat,” if the note is the same pitch as the previous note.

For example, the song Happy Birthday starts as *RUDU, where ‘*’ is the initial note. We Are The World starts as *RDUD.

When considering communication over a discrete-time channel with ISI (i.e. that has memory characterized by \( h[n] \)), we usually think of the input signal \( x[n] \) as being memoryless. In the case of music identification, however, the input signal \( x[n] \) has memory across time, but the channel \( h[n] \) is without ISI.

(a) Suppose we transmit the initial five-note sequence, called \( x_a[1], \ldots, x_a[5] \), for Happy Birthday, such that the starting note is \(-\sqrt{E}\) and the increment for moving up and down is \( 2\sqrt{E} \). Write down \( x_a[1], \ldots, x_a[5] \).

**Solution**

Follows directly that \( x_a[1], \ldots, x_a[5] \) is \((-\sqrt{E}, -\sqrt{E}, \sqrt{E}, -\sqrt{E}, \sqrt{E})\).

(b) Suppose we transmit the initial five-note sequence, called \( x_b[1], \ldots, x_b[5] \), for We Are The World, such that the starting note is \( \sqrt{E} \) and the increment for moving up and down is \( 2\sqrt{E} \). Write down \( x_b[1], \ldots, x_b[5] \).

**Solution**

Follows directly that \( x_b[1], \ldots, x_b[5] \) is \((\sqrt{E}, \sqrt{E}, -\sqrt{E}, \sqrt{E}, -\sqrt{E})\).
(c) A communication system has two possible ringtones, either $x_a[n]$ or $x_b[n]$, used equiprobably, that are transmitted as complete sequences over an AWGN channel with i.i.d. noise $w[n] \sim \mathcal{N}(0, \sigma^2)$ to produce $y[1], y[2], \ldots, y[5]$. Let $\text{SNR} = E/\sigma^2$. What is the optimal decision rule and error probability $P_e^{(2)}$ when we use only the first two symbols $y[1], y[2]$ to decode?

$$P_e^{(2)} = Q(\sqrt{2\text{SNR}})$$

Solution

Since $x_a[1], x_a[2]$ is $(-\sqrt{E}, -\sqrt{E})$ and since $x_b[1], x_b[2]$ is $(\sqrt{E}, \sqrt{E})$, this is just a repetition code over an AWGN channel. The optimal decision rule is:

$$\begin{cases} x_a, & y[1] + y[2] < 0 \\ x_b, & y[1] + y[2] \geq 0. \end{cases}$$

The error probability for two repetitions is $Q(\sqrt{2\text{SNR}})$.

(d) Now fix the ringtone as $x_a[n]$ and consider deciding whether the phone is ringing or producing the all-zero sequence, $x_z[1] = \cdots = x_z[5] = 0$. The phone equiprobably rings or not. Again complete sequences are transmitted over an AWGN channel with i.i.d. noise $w[n] \sim \mathcal{N}(0, \sigma^2)$ to produce $y[1], y[2], \ldots, y[5]$. Let $\text{SNR} = E/\sigma^2$. Before making a decision, we pass $y[n]$ through a linear filter $c[1], \ldots, c[5]$ to produce $\hat{y} = \sum_{i=1}^{5} c[i]y[i]$ and base the decision on it. Find the optimal filter $c[n]$ and error probability $P_e^{(5)}$ when using the optimal decision rule.

$$P_e^{(5)} = Q\left(\frac{5\sqrt{E}}{2\sigma}\right)$$

Solution

The best filter is the matched filter, so $c[i] = x_a[i]$. For simplicity, let us not worry about the magnitude, and just take the signs: $[-1, -1, 1, -1, 1]$.

After applying the matched filter, we have to compare $\hat{y} = 0$ against $\hat{y} = 5\sqrt{E}$. The noise variance we face is $5\sigma^2$. The optimal decision rule is nearest neighbor since equiprobable, and is therefore:

$$\begin{cases} \text{ring,} & \hat{y} > \frac{5}{2}\sqrt{E} \\ \text{no ring,} & \hat{y} \leq \frac{5}{2}\sqrt{E}. \end{cases}$$

The performance of such a scheme is then $Q\left(\frac{5\sqrt{E}}{2\sigma}\right)$.
3. Consider the 3-tap ISI channel with:

\[ y[m] = x[m] + x[m-1] + x[m-2] + w[m]. \]

Only two symbols \( x[1] \) and \( x[2] \) are sent on this channel, i.e., you can assume that \( x[m] = 0 \), for \( m \neq 1, 2 \). Assume that \( x[1] \) and \( x[2] \) are i.i.d. with mean 0 and variance \( E \), and the \( w[m] \) are i.i.d. \( \sim \mathcal{N}(0, \sigma^2) \). We define \( \text{SNR} = E/\sigma^2 \).

The goal is to detect \( x[1] \) from \( y[1], y[2], \) and \( y[3] \), using a linear equalizer based on the sum:

\[ \hat{y}[1] = c_1 y[1] + c_2 y[2] + c_3 y[3]. \]

(a) Find the relationship between \( c_2 \) and \( c_3 \) that any zero-forcing solution must satisfy and write the resulting \( \hat{y}[1] \) in terms of \( c_1 \) and \( c_2 \) only.

**Solution**

\[ \hat{y}[1] = (c_1 + c_2 + c_3) x[1] + (c_2 + c_3) x[2] + (c_1 w_1 + c_2 w_2 + c_3 w_3). \]

Let the coefficient on \( x[2] \) be zero,

\[ c_2 + c_3 = 0 \Rightarrow c_2 = -c_3. \]

\( \hat{y}[1] \) can be reduced to

\[ \hat{y}[1] = c_1 x[1] + (c_1 w_1 + c_2 w_2 - c_2 w_3). \]

(b) Using your solution from part (a), write down the SINR of the zero-forcing solution as a function of \( c_1 \) and \( c_2 \). Then find the \( c_1 \) and \( c_2 \) that maximize this SINR and the corresponding maximum value of the SINR as a function of \( \text{SNR} \).

**Solution**

\[ \text{SINR} = \frac{c_1^2 E}{(c_1^2 + 2c_2^2)\sigma^2} = \frac{c_1^2}{c_1^2 + 2c_2^2} \text{SNR}, \]

which is a decreasing function of \( c_2^2 \). Thus the maximum is achieved when \( c_2 = 0 \).

\[ \text{SINR} = \frac{c_1^2}{c_1^2} \text{SNR} = \text{SNR}. \]
(c) Find the linear transformation of the channel outputs that whitens the interference plus noise terms in the channel outputs.

**Solution**

Let

\[
\begin{align*}
\end{align*}
\]

where

\[
\begin{align*}
\end{align*}
\]

\(z[2]\) and \(z[3]\) are correlated. Let

\[
\begin{align*}
\hat{y}[1] &= y[1] \\
\hat{y}[2] &= y[2] \\
\end{align*}
\]

where \(a\) is determined such that

\[
\]

is uncorrelated with \(\hat{z}[2] = z[2]\).

\[
0 = \text{cov}(\hat{z}[3], \hat{z}[2]) = \text{cov}(z[3], z[2]) - a \cdot \text{cov}(z[2], z[2]),
\]

\[
\Rightarrow a = \frac{\text{cov}(z[3], z[2])}{\text{cov}(z[2], z[2])} = \frac{E}{E + \sigma^2} = \frac{\text{SNR}}{\text{SNR} + 1}.
\]
4. For each of the following statements, mark whether they are true or false. Justify your answer briefly.

(a) The FIR filter response, \( h_0, \ldots, h_{L-1} \) of the wireline channel model depends on the analog quantities: sampling rate \( T \), pulse shaping filter \( g(\cdot) \), and continuous-time wireline channel impulse response \( h(\cdot) \), of which all three can be chosen by the communication engineer.

\[ \text{Mostly TRUE} \]

**Solution** The FIR filter response \( h \) definitely depends on the three things mentioned. Sampling rate and pulse shaping filter can definitely be chosen by the mathematical communication engineer. The CT wireline response can sometimes be chosen by the physical communication engineer, but not always.

(b) Consider communicating on a continuous-time AWGN channel with average power of \( P = 10^{-4} \) Watts, and a noise power spectral density of \( N_0 = 10^{-6} \) Watts/Hz. With sufficiently large bandwidth, it should be possible to send 10,000 bits/s reliably on the channel.

\[ \text{FALSE} \]

**Solution** As the bandwidth \( W \to \infty \), the capacity of the AWGN channel increases to \( \frac{P}{N_0} \log_2 e = 100 \log_2 e \), which is less than 10,000.

(c) Consider a two-tap ISI channel with \( h_0 = 1 \) and \( h_1 = 0.5 \). The naive precoding strategy that sets \( x[m] = d[m] - I[m] \), with \( d[m] = \pm 1 \), has a transmit energy that always remains bounded as \( m \to \infty \), no matter what the realization of the \( d[m] \) sequence is.

\[ \text{TRUE} \]

**Solution** The maximum value of the transmit energy occurs when \( d[m] \) takes the value 1 for odd \( m \) and the value \(-1\) for even \( m \). Even in that case, the transmit energy is given by \( 1 + 0.5 + 0.5^2 + 0.5^3 + \cdots = 2 < \infty \).

(d) The capacity-maximizing power allocation strategy on an OFDM channel equalizes the powers on all of the subchannels.

\[ \text{FALSE} \]

**Solution** The capacity-maximizing power allocation strategy is the “water-filling” strategy, which initially allocates more power to channels that have larger gains.