Name: (in BLOCK CAPITALS) ________________________________

Signature: ________________________________

Instructions
This is a closed-book closed-notes examination except that one side of one 8.5” × 11” sheet of notes is permitted. Tables of integrals, calculators, computers, cell phones, microscopes, etc. are neither needed nor permitted.

SHOW YOUR WORK. If you need additional space, use the back of the previous page. Write your final answers in the spaces provided.

This exam contains four problems on six separate pages

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1. In a certain communication system, the received signal $Y$ has probability density function $f_{Y|X}(y|X = x_1) = \frac{1}{2} \exp(-|y - \mu|)$ if the message $x = x_1$ is transmitted. The received signal $Y$ has probability density function $f_{Y|X}(y|X = x_2) = \frac{1}{2} \exp(-|y + \mu|)$ if the message $x = x_2$ is transmitted. Message $x_1$ is sent 20% of the time and $x_2$ is sent otherwise.

(a) Design the decision rule that minimizes the probability of error in deciding on the transmitted message.

**Solution:** We can sketch out the conditional pdfs to see what is going on, as depicted below, but we should note that the two messages are not sent equiprobably and so the MAP rule (the rule that minimizes error probability) is not just deciding whether positive or negative. We need to work through the likelihood ratio and compare to the threshold determined by the ratio of prior probabilities.

\[
f_{Y|X}(y|x_1) = \begin{cases} \frac{1}{2} \exp(-y + \mu), & \text{if } y \geq \mu, \\ \frac{1}{2} \exp(y - \mu), & \text{if } y < \mu, \end{cases}
\]

whereas

\[
f_{Y|X}(y|x_2) = \begin{cases} \frac{1}{2} \exp(-y - \mu), & \text{if } y > -\mu, \\ \frac{1}{2} \exp(y + \mu), & \text{if } y \leq -\mu, \end{cases}
\]

and thus

\[
\Lambda(y) = \frac{f_{Y|X}(y|x_2)}{f_{Y|X}(y|x_1)} = \begin{cases} \exp(2\mu), & \text{if } y \leq -\mu, \\ \exp(-2y), & \text{if } -\mu < y < \mu, \\ \exp(-2\mu), & \text{if } y \geq \mu, \end{cases}
\]

takes on values in the interval $[\exp(-2\mu), \exp(2\mu)]$.

The MAP rule decides that a 1 was transmitted if $\Lambda(y) > \pi_1/\pi_2$, where for us, $
\pi_1/\pi_2 = 0.2/0.8 = 1/4$. Now, if $1/4 > \exp(2\mu)$, the likelihood ratio can never exceed the threshold and the decision is that a $x_1$ was transmitted regardless of the observed value of $y$. Similarly, if $1/4 < \exp(-2\mu)$, the likelihood ratio always exceeds the threshold and the decision is that a $x_2$ is transmitted regardless of the observed value of $y$. For values of $1/4$ in the interval $(\exp(-2\mu), \exp(2\mu))$, the MAP rule can be expressed as

\[
\text{Decide } x_2 \text{ was transmitted if: } y < -\frac{1}{2} \ln \left( \frac{1}{4} \right) = \ln \sqrt{4} = \ln 2.
\]
(b) Evaluate the average error probability, which we call $P_e^{(\ast)}$.

Solution:
When in the intermediate region, the threshold of the likelihood ratio test is $\theta = \ln 2 \in (0, \mu)$. Thus, we have

\[
P_{e,1} = \Pr\{ y < \theta \mid x_1 \text{ transmitted} \} = \int_{-\infty}^{\theta} f_{Y \mid X}(y \mid x_1) \, dy
\]
\[
= \int_{-\infty}^{\theta} \frac{1}{2} \exp(y - \mu) \, dy = \frac{1}{2} \exp(\theta - \mu) = \frac{1}{2} \exp(\ln 2) \exp(-\mu) = \exp(-\mu),
\]

\[
P_{e,2} = \Pr\{ y > \theta \mid x_2 \text{ transmitted} \} = \int_{\theta}^{\infty} f_{Y \mid X}(y \mid x_2) \, dy
\]
\[
= \int_{\theta}^{\infty} \frac{1}{2} \exp(-y - \mu) \, dy = \frac{1}{2} \exp(-\theta - \mu) = \frac{1}{4} \exp(-\mu).
\]

Now, the average error probability is just

\[
P_e^{(\ast)} = \pi_1 P_{e,1} + \pi_2 P_{e,2} = \frac{1}{5} \exp(-\mu) + \frac{4}{5} \cdot \frac{1}{4} \exp(-\mu) = \frac{2}{5} \exp(-\mu)
\]

When in one of the regimes where the measurement is irrelevant, an error occurs only when the undecided message was transmitted.

(c) Design the maximum likelihood decision rule for deciding on the transmitted message and evaluate the error probability, which we call $P_e^{(\text{ML})}$.

Solution:
Clearly the ML decision rule is to decide based on whether $y$ is positive or negative.

\[
P_{e,1} = \Pr\{ y < 0 \mid x_1 \text{ transmitted} \} = \int_{-\infty}^{0} f_{Y \mid X}(y \mid x_1) \, dy
\]
\[
= \int_{-\infty}^{0} \frac{1}{2} \exp(y - \mu) \, dy = \frac{1}{2} \exp(-\mu)
\]

and likewise by symmetry

\[
P_{e,2} = \frac{1}{2} \exp(-\mu).
\]

Since $P_{e,1} = P_{e,2} = \frac{1}{2} \exp(-\mu)$, this is also $P_e^{(\text{ML})}$.

(d) Is $P_e^{(\ast)} \geq P_e^{(\text{ML})}$?

Solution:
NO, by comparison. The MAP decoder always outperforms the ML decoder.
2. Let us send one information bit on a discrete-time AWGN channel using 3 time instants:

\[ y[m] = x[m] + w[m], m = 1, 2, 3 \]

where \( \{w[m]\} \) are independent \( \mathcal{N}(0, \sigma^2) \) random variables. We transmit voltage vectors

\[
\vec{v}_0 = [-\sqrt{E} \quad \sqrt{E} \quad 0] \quad \text{and} \quad \vec{v}_1 = [\sqrt{E} \quad -\sqrt{E} \quad 0]
\]

where \( \vec{x} = \vec{v}_0 \) is transmitted for the first message and \( \vec{x} = \vec{v}_1 \) is transmitted for the second message; messages are equiprobable. The optimal minimum distance decoding rule is used.

(a) Expressed in terms of elementary functions or the \( Q \) function, what is the average probability of error \( P_e \) as a function of \( \text{SNR} = E/\sigma^2 \)?

\[
P_e(\text{SNR}) = Q(\sqrt{2\text{SNR}}).
\]

**Solution:** The last symbol is irrelevant, so we do not use it in optimal decoding or error probability computation. Due to symmetry, to compute reliability we can permute the second symbol in the signals, to get back to a \( (n = 2) \)-symbol repetition scheme for sending one information bit. We know that in the presence of AWGN noise with \( \text{SNR} = E/\sigma^2 \), the performance is just

\[
P_e(\text{SNR}) = Q(\sqrt{n\text{SNR}}) = Q(\sqrt{2\text{SNR}}).
\]

(b) What is the rate of communication (in units of bits per channel use)?

\[
R = \frac{1}{3}
\]

**Solution:** Three channel usages are required for this signalling scheme. One information bit is transmitted, so the rate is: \( R = 1/3 \) bits per channel use.
(c) What is the average energy efficiency of communication (in units of energy per bit)?

\[ E_b = \]

**Solution:** Each of the first two symbols uses energy \( E \), and the third symbol does not use energy. One information bit is transmitted, so the energy efficiency is \( 2E \) units of energy per bit: \( E_b = 2 \).

(d) Describe a simple way to change the communication scheme to improve the rate, while keeping the average error probability and energy efficiency constant.

**Solution:** We can simply not send the third symbol. This will improve the rate to \( 1/2 \) and keep the same average error probability and energy efficiency, since the third symbol in the current scheme neither differentiates the two messages nor uses energy.
3. Consider the (7,4) Hamming code with generator matrix

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]

If we transmit a codeword \( \vec{c} \) from the code over a binary symmetric channel with crossover probability 0.1 and receive the following sequence \([1 1 1 0 0 1]\), what is the maximum likelihood estimate of \( \vec{c} \)?

\[\hat{\vec{c}} = \]

**Solution:** For binary Hamming codes, if \( G = [I|A] \), then \( H = [A^T|I] \), so

\[
H = \begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

We know \( Hc = 0 \), therefore we get the following equalities:

\[
c_1 + c_2 + c_3 + c_5 = 0 \\
c_2 + c_3 + c_4 + c_6 = 0 \\
c_1 + c_2 + c_4 + c_7 = 0
\]

We can observe that the first and second parity equations are not satisfied, whereas the third one is. Further, we can observe that the first and second parity equations would become satisfied if \( c_3 \) were 0 instead of 1, e.g. by drawing a Venn diagram. (\( c_2 \) is not flipped, since it participates in the third parity equation.)

Hence, the ML estimate is made by flipping \( c_3 \), yielding \( \hat{\vec{c}} = [1 1 0 1 0 0 1] \).
4. For each of the following statements, mark whether they are true or false. Justify your answer briefly.

(a) Let the random variable $W$ be the difference of two independent Gaussian noises with the same mean and variance. Then $W$ is distributed according to the Rayleigh distribution and has twice the variance as the original Gaussian noises.

Solution: FALSE, the difference of Gaussian noises is Gaussian, due to $\alpha$-stability.

(b) Consider communication over a discrete-time AWGN channel using a repetition code, with received voltages $y[m] = x + w[m], m = 1, \ldots, n$. Then, the sum of received voltages $\sum_{m=1}^{n} y[m]$ is a sufficient statistic to perform maximum likelihood decoding.

Solution: TRUE, the sum of the voltages is indeed a sufficient statistics for maximum likelihood decoding, as follows from a vector space view.

(c) Consider a binary erasure channel with erasure probability $p$. Then, there exists a nonlinear code that can be used to communicate arbitrarily reliably at information rates greater than $1 - p$.

Solution: FALSE, as $1 - p$ is the capacity of a binary erasure channel, and it is impossible to communicate reliably at rates greater than capacity.

(d) For a continuous-time AWGN channel, when the bandwidth is large, the capacity is approximately proportional to the total power $\bar{P}$ received over the whole band.

Solution: TRUE, as follows from the capacity approximation in the power-limited regime.