ECE 361: First Exam
Tuesday, 1 March 2015, 9:30 am – 10:50 am

Name: (in BLOCK CAPITALS) ________________________________

Signature: ________________________________

Instructions
This is a closed-book closed-notes examination except that one side of one 8.5” × 11” sheet of notes is permitted. Tables of integrals, calculators, computers, cell phones, microscopes, etc. are neither needed nor permitted.

SHOW YOUR WORK. If you need additional space, use the back of the previous page. Write your final answers in the spaces provided.

This exam contains four problems on seven separate pages

<table>
<thead>
<tr>
<th>Grading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 24 points ____________</td>
</tr>
<tr>
<td>2. 28 points ____________</td>
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<tr>
<td>3. 24 points ____________</td>
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<td>4. 24 points ____________</td>
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<tr>
<td>Total (100 points) ____________</td>
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</tbody>
</table>
1. Consider an information source that is generating the letters \{\alpha, \beta, \gamma, \delta, \varepsilon\} with the respective probabilities \{1/12, 1/6, 1/6, 1/4, 1/3\}.

(a) Someone is suggesting the following binary source code mapping:

<table>
<thead>
<tr>
<th>symbol</th>
<th>codeword</th>
</tr>
</thead>
<tbody>
<tr>
<td>\alpha</td>
<td>0</td>
</tr>
<tr>
<td>\beta</td>
<td>1</td>
</tr>
<tr>
<td>\gamma</td>
<td>00</td>
</tr>
<tr>
<td>\delta</td>
<td>01</td>
</tr>
<tr>
<td>\varepsilon</td>
<td>11</td>
</tr>
</tbody>
</table>

Find the average codeword length, \(\bar{\ell}(a)\).

\[
\bar{\ell}(a) = \frac{21}{12}.
\]

Solution
The average codeword length is given as follows:

\[
\bar{\ell}(a) = \frac{1}{12} + \frac{1}{6} + 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{3} = \frac{21}{12}.
\]

(b) Using the same set of codewords as in part (a), find an alternate mapping from symbols to codewords that minimizes the average codeword length. Find this average codeword length, \(\bar{\ell}(b)\).

\[
\bar{\ell}(b) = \frac{17}{12}.
\]

Solution
We can draw on the Morse principle, that shortest codewords should go with the most probable symbols to get the following mapping.

<table>
<thead>
<tr>
<th>symbol</th>
<th>codeword</th>
</tr>
</thead>
<tbody>
<tr>
<td>\varepsilon</td>
<td>0</td>
</tr>
<tr>
<td>\delta</td>
<td>1</td>
</tr>
<tr>
<td>\gamma</td>
<td>00</td>
</tr>
<tr>
<td>\beta</td>
<td>01</td>
</tr>
<tr>
<td>\alpha</td>
<td>11</td>
</tr>
</tbody>
</table>

The average codeword length is then given as follows:

\[
\bar{\ell}(b) = \frac{1}{3} + \frac{1}{4} + 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{12} = \frac{17}{12}.
\]
(c) Are the set of codewords in part (a) uniquely decodable?

YES or NO? NO

Solution
According to the Kraft-McMillan theorem, the codewords in a uniquely decodable code must satisfy the Kraft inequality. Checking that here, we observe that the Kraft inequality is not satisfied, so no.

\[ 2^{-1} + 2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} = \frac{7}{4} > 1. \]

(d) Construct a binary Huffman code for this source and find the average codeword length, \( \bar{\ell}_{\text{Huff}} \).

\[ \bar{\ell}_{\text{Huff}} = \frac{9}{4} \]

Solution
An example of a Huffman code is as follows, as obtained through the tree construction.

<table>
<thead>
<tr>
<th>symbol</th>
<th>codeword</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>000</td>
</tr>
<tr>
<td>( \beta )</td>
<td>001</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>01</td>
</tr>
<tr>
<td>( \delta )</td>
<td>10</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>11</td>
</tr>
</tbody>
</table>

The average codeword length is then given as follows:

\[
\bar{\ell}_{\text{Huff}} = 3 \cdot \frac{1}{12} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{3}
\]

\[
= \frac{9}{4}.
\]
2. The probability density function of the Kumaraswamy distribution is

\[ p_W(w) = abw^{a-1}(1 - w^a)^{b-1}, \]  

where \( w \in [0, 1] \), and where \( a \) and \( b \) are non-negative shape parameters. In this problem, we consider communication over a discrete-time additive white Kumaraswamy noise (AWKN) channel with \( a = 2 \) and \( b = 2 \). Note that this noise is supported on the interval \([0, 1]\) and is not symmetric around 1/2.

(a) Suppose we signal using an alphabet \( x \in \{0, 2\} \), where the probability of \( x = 0 \) is 0.3 and the probability of \( x = 2 \) is 0.7. Design a decision rule that minimizes the probability of error in deciding on the transmitted message.

Solution

We notice that the conditional probability distributions for the two messages do not overlap, as one has support on the interval \([0, 1]\) and the other on the interval \([2, 3]\). Thus any threshold-based test with a threshold in the non-overlapping region would minimize error probability. As a particular example, letting the channel output be \( y \) and the decision \( \hat{x} \), an optimal rule is:

\[
\begin{align*}
\hat{x} = 0 & \quad \text{if } y \leq 1.5, \\
\hat{x} = 2 & \quad \text{if } y > 1.5.
\end{align*}
\]

(b) Evaluate the average error probability, which we call \( P_e^{(b)} \).

\[ P_e^{(b)} = 0 \]

Solution

Clearly there is no error, since there is no confusion.
(c) Suppose we signal using an alphabet \( x \in \{0, \frac{1}{2}\} \), where the probability of \( x = 0 \) is 0.5 and the probability of \( x = \frac{1}{2} \) is 0.5. Design the decision rule that minimizes the probability of error in deciding on the transmitted message.

**Solution**

Under one hypothesis, we have a conditional output distribution of \( 4y(1 - y^2) \) and in the other, we have a conditional output distribution of \( 4(y - \frac{1}{2})(1 - (y - \frac{1}{2})^2) \). Since the two messages are equiprobable, we just need to compare the likelihoods. Moreover, we can sketch and observe we just need to find the crossing point to get a threshold test. Let us equate to find the threshold.

\[
4y(1 - y^2) = 4(y - \frac{1}{2})(1 - (y - \frac{1}{2})^2) \\
y(1 - y^2) = (y - \frac{1}{2})(1 - (y^2 - y + \frac{1}{2} )) \\
y - y^3 = (y - \frac{1}{2})(-y^2 + y + \frac{3}{4}) \\
y - y^3 = -y^3 + y^2 + \frac{3}{4}y + \frac{1}{2}y^2 - \frac{1}{2}y - \frac{3}{8} \\
y = \frac{3}{2}y^2 + \frac{1}{4}y - \frac{3}{8} \\
o = \frac{3}{2}y^2 - \frac{3}{4}y - \frac{3}{8} \\
o = 3y^2 - \frac{3}{2}y - \frac{3}{4} \\
o = y^2 - \frac{3}{2}y - \frac{1}{4}.
\]

Now using the quadratic formula, we get the following roots of the equation.

\[
\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \cdot \frac{-1}{4}} = \frac{1}{2} \left( \frac{1}{2} \pm \frac{\sqrt{5}}{2} \right)
\]

Since the noise and signaling are positive-valued, we care about the positive root for our threshold. This is

\[
\frac{1}{2} \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) = \frac{1 + \sqrt{5}}{4}.
\]

Thus our decision rule is

\[
y \leq 0 \quad \text{1} + \frac{\sqrt{5}}{4} \\
y \geq \frac{1}{2} \quad \frac{1 + \sqrt{5}}{4}.
\]

(d) Let us call the average probability of error in this case to be \( P_e^{(d)} \). Is \( P_e^{(d)} \geq P_e^{(b)} \)?

**Solution**

Clearly there will be some positive error probability, since there is overlap. This is more than zero.
3. Consider a discrete-time AWGN channel with noise \( N(0, 1) \). We will perform sequential communication, where two symbols are used equiprobably: \( \{-\sqrt{E}, \sqrt{E}\} \), together with erasures demodulation where the boundaries of the erasure region are \(-\sqrt{E}/3, \sqrt{E}/3\).

(a) Expressed in terms of elementary functions or the \( Q \) function, what is the average probability of erasure \( p \)?

\[
p = Q \left( \frac{2\sqrt{3}}{3} \right) - Q \left( \frac{4\sqrt{3}}{3} \right)
\]

**Solution**
Due to symmetry, we need only worry about one of the messages, say the negative one. We already have variance 1, but let us shift things to the right by \( \sqrt{3} \) so we have mean 0. After shifting, the erasure region is from \( \sqrt{E} - \sqrt{E}/3 \) to \( \sqrt{E} + \sqrt{E}/3 \). Thus the probability will be the difference of two \( Q \) functions. In particular, we subtract the long tail from the intermediate region to get

\[
p = Q \left( \sqrt{E} - \sqrt{E}/3 \right) - Q \left( \sqrt{E} + \sqrt{E}/3 \right)
= Q \left( \frac{2\sqrt{3}}{3} \right) - Q \left( \frac{4\sqrt{3}}{3} \right).
\]

(b) Expressed in terms of elementary functions or the \( Q \) function, what is the average probability of error \( P_e \) (which we sometimes declare to be negligible)?

\[
P_e = Q \left( \frac{4\sqrt{3}}{3} \right)
\]

**Solution**
This is the long tail that we just subtracted:

\[
P_e = Q \left( \frac{4\sqrt{3}}{3} \right).
\]
(c) What is the rate of communication (in units of bits per channel use)?

\[ R = 1 \]

Solution

In every channel use, we are sending one of two equiprobable messages, and so 1 bit. The rate is therefore 1 bit/channel use.

(d) What is the average energy efficiency of communication (in units of energy per bit)?

\[ E_b = E \]

Solution

Each of the two signals uses energy \( E \), since they are \( \pm \sqrt{E} \). There is one bit that is transmitted. Hence, the energy efficiency is \( E \) energy per bit.
4. For each of the following statements, mark whether they are true or false. Justify your answer briefly.

(a) Let the random variable \( W \) be the sum-of-squares of two dependent Gaussian noises with the same mean and variance. Then \( W \) is distributed according to the Gaussian distribution with twice the variance as the original noises.

\[ \text{TRUE or FALSE? FALSE} \]

Solution
The square of a Gaussian random variable is positive-valued, and can therefore not be Gaussian itself.

(b) Consider communication over a discrete-time AWGN channel using a repetition code, with received voltages \( y[m] = x + w[m], m = 1, \ldots, n \). Then, the midrange of received voltages \( \min\{y[m]\} + \max\{y[m]\} \) is a sufficient statistic to perform maximum likelihood decoding.

\[ \text{TRUE or FALSE? FALSE} \]

Solution
The sum or the mean would be a sufficient statistic, but the midrange is not. It loses useful information.

(c) The International Standard Book Number (ISBN) system assigns a unique 10-digit number to each book, where the first 9 digits are information symbols whereas the tenth digit is a checksum introduced for error detection/correction. The number of information bits that can be encoded in this system is \( 9 \log_2 10 \) bits, and the rate of the error-correcting code is \( \frac{\log_2 9}{\log_2 10} \).

\[ \text{TRUE or FALSE? FALSE} \]

Solution
The number of bits is correct, but the rate is not correct. That is \( 9/10 \).

(d) Suppose a large linear code has generator matrix with linearly independent rows (is full-rank) and has design rate \( 6/7 \). If this code is used to communicate over a binary erasure channel with erasure probability \( 1/10 \), then with high probability communication can be carried out with perfect reliability.

\[ \text{TRUE or FALSE? TRUE} \]

Solution
The rate, \( 6/7 \), is less than the capacity, \( 9/10 \), so yes, according to Shannon’s noisy channel coding theorem.