1 Special relativity for ECE students

ECE Illinois students are not required to take PHYS 225 where the topic of special relativity is covered in full detail. Here we describe an alternate and succinct coverage of the elements of special relativity within ECE 350 as an extension of the existing unit on Doppler effect. I feel that this mode of introduction to special relativity is closer in spirit to the title *Albert Einstein* chose for his 1905 relativity paper: "On the Electrodynamics of Moving Bodies".

Erhan Kudeki, 10/26/2014; last revision 04/27/22

1.1 Preliminaries:

Consider a TEM plane wave propagating in the x-direction with field components varying as

$$\cos(\omega t - kx) \tag{1}$$

which I (*Erhan* ;-) monitor with spaced antennas and connected scopes stationary in my **lab frame** to determine the oscillation period $T = 2\pi/\omega$ and wavelength $\lambda = 2\pi/k$ such that $\omega/k = c$.

Albert is traveling in my lab in the x-direction with a speed v and he is also monitoring the very same wave with similar equipment which are stationary in his reference frame, which we will call the **primed frame,** where the field components vary as

$$\cos(\omega' t' - k' x') \tag{2}$$

in terms of primed coordinates x' and t' — note that $\omega'/k' = c$ also, since Maxwell's equations are equally valid in all frames and predict the same travel speed for their TEM wave solutions.

According to "Galilean transformations" of classical mechanics relating the space and time coordinates of objects monitored in the two frames

$$x' = x - vt \tag{3}$$
$$t' = t, \tag{4}$$

but we can easily show the incompatibility of these relations with the invariance of the TEM wave speed in all reference frames:

Galilean transformations are incompatible with frame independent light–speed:

Let $u \equiv dx/dt$ and $u' \equiv dx'/dt'$ denote the travel speeds of some particle in the lab and primed frames, respectively. Differentiating (3) with respect to t' and using (4) we find

$$u' = u - v, \tag{5}$$

which leads to $u' = c - v$ with $u = c$ for a light particle (photon), which is at odds with the fact that $u' = c$ also when a light particle is considered. Hence, the Galilean transformation rules (3-4) will need to be revised for applications with fast moving particles.

1.2 Non-relativistic Doppler shift formulae:

Evaluating and equating the waveforms (1) and (2) at the instantaneous location $x = vt$ and $x' = 0$ of Albert, respectively, and using (4), we obtain

$$\cos(\omega t - kvt) = \cos((\omega - kv)t) = \cos(\omega' t') = \cos(\omega' t) \tag{6}$$

and hence conclude that

$$\omega' = \omega - kv, \tag{7}$$

which is the *non-relativistic* **Doppler shift** equation for the case of **red shift** — Albert is moving in the direction of the propagating wave. Likewise, for the case of **blue shift** — Albert moving against the propagating wave — we have

$$\omega' = \omega + kv. \tag{8}$$

We will refer to (7) and (8) as **one-way Doppler shift** equations since we will also have **two-way Doppler shift** equations that can be expressed succinctly for red and blue shift cases as

$$\omega'' = \omega \mp 2kv, \tag{9}$$

where ω'' denotes the frequency of a reflected version of (1) from a mirror carried by Albert and detected in the lab frame.

Note that (9) is obtained by applying the $\omega \mp kv$ transformation twice in succession, first to go from ω to ω' and next from ω' to ω'', under the logic that *transformation rules must be* invariant (just like Maxwell's equations) in all reference frames.

We will find out that the one-way and two-way Doppler formulae stated above are only valid for $|v| \ll c$, as limiting cases of more rigorously derived Doppler formulae in the following section. The shortcomings of the non-relativistic Doppler formulae can be traced to our use of (4) in the derivation of (7) and (8), in addition to the use of k in place of k' in the derivation of (9). We will discover the relativistically valid counterpart of (4) later on after we derive the relativistically valid Doppler formulae below using the Maxwell's equations.

1.3 Relativistic Doppler shift formulae:

Let us assume that the relativistically correct red- and blue-shifted one-way Doppler formulae are given as

$$\omega' = \omega\sqrt{\frac{1 \mp \frac{v}{c}}{1 \pm \frac{v}{c}}}. \tag{10}$$

With (10) and using the principle of the invariance of transformation rules between different reference frames, the corresponding two-way Doppler formulae is obtained as

$$\omega'' = \omega'\sqrt{\frac{1 \mp \frac{v}{c}}{1 \pm \frac{v}{c}}} = \omega\frac{1 \mp \frac{v}{c}}{1 \pm \frac{v}{c}}. \tag{11}$$

It is now sufficient to derive (11) from the frame invariant Maxwell's equations — as shown next — in order to prove the validity of both (11) and (10):

Proof of 2-way Doppler formula: Consider the TEM wave field $\propto \cos(\omega t - kx)$ observed in the lab frame to be reflected from a mirror carried by Albert at $x' = 0$ coinciding with $x = vt$ in the lab frame so that the total field in the lab frame to the left of the mirror varies as a superposition

$$\cos(\omega t - kx) + R\cos(\omega'' t + k'' x + \phi) \tag{12}$$

that will satisfy the Maxwell's equations in the lab frame so long as $\omega''/k'' = c$. In this expression think of R and ϕ as the magnitude and angle of a reflection coefficient $\Gamma = Re^{j\phi}$ describing the reflection taking place at $x = vt$. Now, the field, evaluated at $x = vt$, at the location of the reflecting surface, will be

$$\cos(\omega t - kvt) + R\cos(\omega'' t + k'' vt + \phi) = \cos(\omega(1 - \frac{v}{c})t) + R\cos(\omega''(1 + \frac{v}{c})t + \phi), \tag{13}$$

which is required to *match* as a boundary condition the *monochromatic* field variation of a form $\propto \cos(\omega' t' + \phi')$ observed at $x' = 0$ in Albert's frame. This is possible, that is (13) is monochromatic, *if and only if*

$$\omega''(1 + \frac{v}{c})t = \omega(1 - \frac{v}{c})t, \tag{14}$$

which leads to the two-way red-shifted Doppler formula

$$\omega'' = \omega \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}$$

that is compatible with (11). The blue shifted version follows by replacing v with $-v$. Therefore we have proven the validity of the relativistic Doppler formulae (11) and (10) based on the validity and frame invariance of Maxwell's equations and Doppler frequency transformation rules. See Figure 1 and caption for an alternate and graphical derivation of (11).

Correspondence relation: While the relativistic (11) and (10) Doppler formulae are valid for all possible $v \leq c$, non-relativistic formulae correspond to their $|v|/c \ll 1$ limit. To verify this correspondence relation we note that

$$\omega' = \omega\sqrt{\frac{1 \mp \frac{v}{c}}{1 \pm \frac{v}{c}}} \approx \omega\sqrt{(1 \mp \frac{v}{c})^2} = \omega(1 \mp \frac{v}{c}) = \omega \mp kv$$

after using $k = \omega/c$ and series expansion formula $(1 + a)^p \approx 1 + pa$ valid for small a.

1.4 Relative velocities:

An object is moving with speed u in x direction in the lab frame. What is the speed u' of the same object in the primed frame?

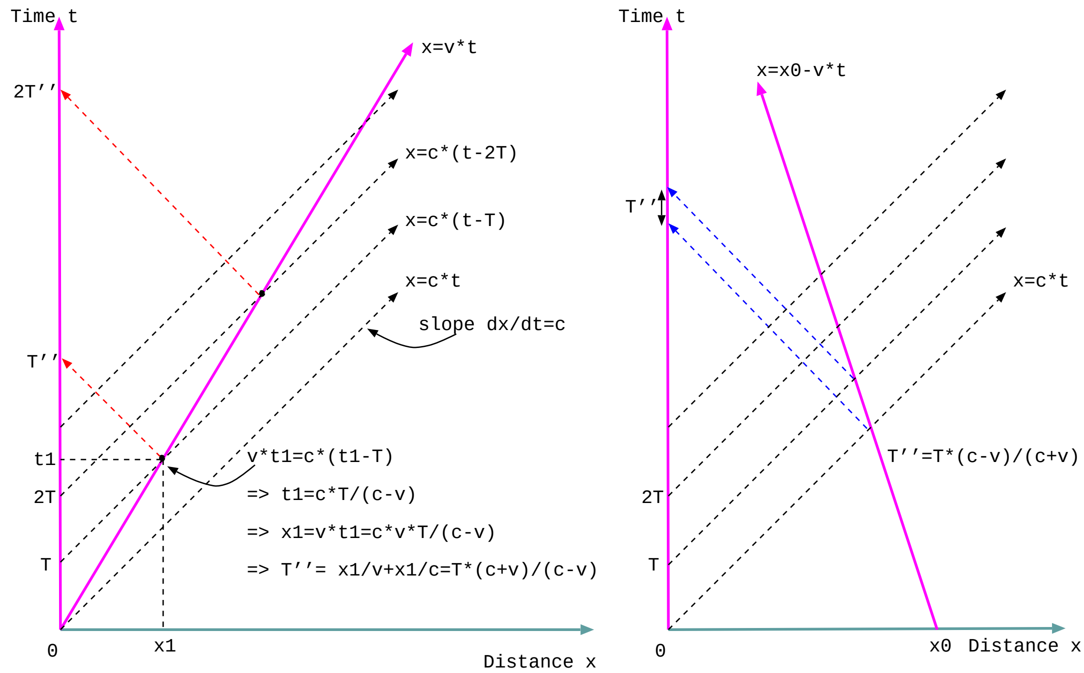

Figure 1: Trajectory plots in the left panel of a moving reflector seen at $x = vt$ in the lab frame and successive crests of a forward propagating TEM wave with a period T encountering the reflector (first encounter at $t = t_1$) to become the crests of a backward propagating wave with the same speed and a redshifted period $T'' = T\frac{1+v/c}{1-v/c}$ seen by a stationary observer in the lab frame (e.g., at x=0). The corresponding redshifted frequency $\omega'' = \omega\frac{1-v/c}{1+v/c}$ (2-way relativistic Doppler formula) follows from $T = 2\pi/\omega$ and $T'' = 2\pi/\omega''$. The right panel illustrates the geometry in the case of blueshift caused by a reflector moving towards the wave source on left hand side.

Classically the answer to the above question would be

$$u' = u - v, \tag{15}$$

but, as we have already seen, this expression gives a non physical result if $u = c$ corresponding to the case of a light particle — it is not consistent with the full set of implications of Maxwell's equations.

We will now derive the relativistically valid version of (15) from relativistic Doppler shift formula (10).

Derivation: Assume that an antenna fixed at the origin of the lab frame is used to transmit a TEM wave of some frequency $f = \omega/2\pi$. The same wave will be detected in the primed frame and in the frame of the object with speed u with Doppler shifted frequencies of

$$f_v = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \text{ and } f_u = f\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}}, \tag{16}$$

respectively. However, we can also express f_u as a Doppler shifted version of f_v as in

$$f_u = f_v\sqrt{\frac{1-\frac{u'}{c}}{1+\frac{u'}{c}}} = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\sqrt{\frac{1-\frac{u'}{c}}{1+\frac{u'}{c}}} \tag{17}$$

where u' is the relative speed of the object in the primed frame. Equating f_u's from (16) and (17) we get

$$\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\sqrt{\frac{1-\frac{u'}{c}}{1+\frac{u'}{c}}} \;\Rightarrow\; \frac{1-\frac{u}{c}}{1+\frac{u}{c}} = \frac{1-\frac{v}{c}}{1+\frac{v}{c}}\frac{1-\frac{u'}{c}}{1+\frac{u'}{c}} = \frac{1-\frac{v+u'}{c}+\frac{vu'}{c^2}}{1+\frac{v+u'}{c}+\frac{vu'}{c^2}},$$

from which

$$(1-\frac{u}{c})(1+\frac{v+u'}{c}+\frac{vu'}{c^2}) = (1+\frac{u}{c})(1-\frac{v+u'}{c}+\frac{vu'}{c^2}) \;\Rightarrow\; v+u'-u-\frac{uvu'}{c^2} = 0$$

leading to relativistic velocity relations

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} \text{ and } u = \frac{u'+v}{1+\frac{u'v}{c^2}} \tag{18}$$

Note that relativistic velocity formula (18) for u' reduces to (15), as it should, in the limit of $|u| \ll c$ and $|v| \ll c$. Also, with $u = c$, it yields

$$u' = \frac{c-v}{1-\frac{cv}{c^2}} = \frac{c-v}{1-\frac{v}{c}} = c,$$

in compliance with the invariance of the speed of light in all reference frames.

Results (18), in effect, establish the speed of light, c, as a *universal speed limit* or the "speed limit of the universe" to be obeyed by all particles — no particle transiting Earth with a (relative) velocity exceeding c has ever been observed!

Illustrative example: A pair of space ships blast away from Earth with speed $u = 0.6c$ in diametrically opposite directions. One of the space ships transmits a 4 GHz signal. (a) What will be the Doppler shifted frequencies detected on Earth and by the second space ship, and (b) What are the relative speeds of the two space ships (with respect to one another)?

Solution:

(a) 4 GHz signal will be detected in Earth frame at

$$f_E = f\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}} = 4\sqrt{\frac{1-0.6}{1+0.6}} = 2\,\text{GHz}.$$

In the frame of the second space ship the Doppler shifted frequency will be

$$f_{S2} = f_E\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}} = 2\sqrt{\frac{1-0.6}{1+0.6}} = 1\,\text{GHz}.$$

(b) The relative speeds of the two space ships would be $1.2c$ classically which is of course an incorrect result. To obtain the relativistically correct result use $u = 0.6c$ and $v = -u$ in (18) relegating space ship 1 into the role of the primed frame. It then follows that the relative speed

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}. = \frac{u + u}{1 + \frac{uu}{c^2}} = \frac{1.2c}{1 + 0.6^2} = \frac{1.2}{1.36}c = \frac{15}{17}c$$

in compliance with the universal speed limit c.

1.5 Time dilation:

Returning to the proof of the relativistic Doppler formula, and combining (13) and (14), we have a monochromatic field variation

$$\cos(\omega(1 - \frac{v}{c})t) + R\cos(\omega(1 - \frac{v}{c})t + \phi) \tag{19}$$

at $x = vt$, matching the primed-frame monochromatic field variation at $x' = 0$

$$\propto \cos(\omega' t' + \phi') = \cos(\omega\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}t' + \phi'). \tag{20}$$

With matching variations in (19) and (20) we have

$$\omega\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}t' = \omega(1 - \frac{v}{c})t,$$

leading to

$$t' = \sqrt{1 - \frac{v^2}{c^2}}t, \tag{21}$$

a result known as **time dilation** equation — since $\sqrt{1 - \frac{v^2}{c^2}} < 1$, this result would indicate that $t' < t$, meaning that time t' of the primed frame must be "passing more slowly" than time t of the lab frame.

Time dilation is a surprising consequence of the "phase invariance" of waves between reference frames in relative motion, indicating that "moving clocks (or hearts)" tick more slowly than "stationary clocks (or hearts)". Moreover, time dilation is a *reciprocal* effect as a consequence of the *relativity of motion* — every clock is "in motion" relative to clocks in other reference frames and thus should be seen "ticking slower" from their perspectives. This very counterintuitive relativistic effect can be verified/confirmed by comparing the readings t' on a moving clock passing by a network of stationary clocks, all keeping the same time t, within the lab frame[1]. Because such confirmations have been

[1] See Figure 2 and its caption for an alternative way of confirming (21) based on relativistic Doppler calculations.

made experimentally (more on this later) and because (21) is "at odds" with the Galilean transformation equation (4) unless $v^2/c^2 \ll 1$, accurate descriptions and models involving fast moving particles require the use of a relativistic framework in place of Newtonian mechanics based on Galilean transformations.

1.6 Lorentz contraction

If time passes more slowly in the moving primed frame as compared to the lab frame as described by *time-dilation* relation

$$t' = \sqrt{1 - \frac{v^2}{c^2}}\,t,$$

then "path lengths" x belonging to the lab frame should "shrink" or "contract" to lengths

$$x' = \sqrt{1 - \frac{v^2}{c^2}}\,x \tag{22}$$

as seen (and measured) in the primed frame — this is true because observers of both frames see one another moving with the same speed v (in opposing directions, of course), given by $\frac{x}{t}$ *as well as* $\frac{x'}{t'}$, in terms of transit time t of Albert across a stationary path of length x (in the frame where Albert is moving), as well as the transit time t' of the same path with a length x' length passing by Albert in Alberts's own reference frame. Hence $v = \frac{x'}{t'} = \frac{x}{t}$, implying that $\frac{x'}{x} = \frac{t'}{t} = \sqrt{1 - \frac{v^2}{c^2}}$, leading to (22) — it is only by having x' shorter than x by the same factor as t' is shorter than t that we get the transit times $t = x/v$ and $t' = x'/v$ to come to agree with (21)!

This is the *Lorentz contraction* effect, a natural companion to *time dilation*, both being very counterintuitive consequences of phase invariance between reference frames:

Moving rulers are contracted while moving clocks run slower compared to their stationary duplicates!!!

Or, equivalently, rulers have their *proper* lengths (i.e., un-contracted, longest) and clocks have their *proper* ticking intervals (i.e., non-dilated, shortest) only in their own reference frames (where they are stationary).

Experimental verification of time-dilation and Lorentz contraction: When cosmic rays collide with atmospheric molecules, high energy *muons* moving with speeds of $\sim 0.98c$ are produced. In its rest frame a muon — a heavier version of electron — will decay to an electron and a pair of neutrinos after a half-life of 1.56 μs. The transit time of a muon produced at 10 km altitude to reach Earth's surface is at a minimum about

$$\frac{10^4\ \text{m}}{0.98 \times 3 \times 10^8\ \text{m/s}} \approx 34\,\mu\text{s}.$$

Since this transit time is more than 20 muon half-lives one would think there should be virtually no muon reaching Earth's surface from the upper atmosphere. Contrary to this, many muons are observed to reach the surface of Earth. The explanation of this[2] provides a verification of time-dilation and Lorentz contraction concepts:

1. When we transform the 34 μs flight time of the muons across the 10 km distance seen as in the *Earth frame* to the *muon frame,* we find out that the flight time only lasts

$$34\,\mu\text{s}\sqrt{1-0.98^2} \approx 6.76\,\mu\text{s}$$

in the muon frame, which is only 4.33 times the muon half-life of 1.56 μs, explaining why many muons are detected after their 34 μs flight as seen in the Earth frame. One can alternatively say that 1.56 μs long muon half-life is dilated to

$$\frac{1.56\,\mu\text{s}}{\sqrt{1-0.98^2}} = 7.84\,\mu\text{s}$$

in the Earth frame, once again leading to same conclusion.

2. The fact that many muons will succeed in reaching Earth's surface (at a rate compatible with the longer half-life of 7.84 μs — see above), can be explained from the point of view of the moving muons (decaying with 1.56 μs proper half-lives) in terms of Lorentz contraction: The proper length of 10 km of the Earth frame distance down to ground is seen in the muon frame as a contracted length of

$$10\,\text{km}\sqrt{1-\frac{v^2}{c^2}} = 10\,\text{km}\sqrt{1-0.98^2} \approx 1.99\,\text{km},$$

which can be transited in

$$\frac{1.99\,\text{km}}{0.98\times 3\times 10^8\,\text{m/s}} \approx 6.76\,\mu\text{s}$$

instead of 34 μs; the fact that the observed muon flux on Earth is compatible with 6.76 μs transit time as opposed to 34 μs is a resounding verification of Lorentz contraction.

1.7 Twin Paradox and time machines

Every process - like aging, like radioactive decay, like the swinging of a simple pendulum - occurs at its own natural rate in its own reference frame but "appears to be" occurring at a slowed rate when viewed from other reference frames[3] — this is the essence of *time dilation*, a mutual/reciprocal effect which has, of course, no impact on what "goes on" in each and every reference frame *per se*.

The impact of time dilation is revealed and *matters* when **observers (or processes) of a given reference frame** "depart" to separate reference frames in relative motion before moving back once more into the same reference frame. Depending on the details of

[2] See http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/muon.html.

[3] using a "latticework" of stationary and synchronized clocks imagined to exist in those frames

how this happens, one of the observers (or processes) will have aged less than the other, having "time travelled", effectively, into the future of the other observer (or process).

For instance, if **Tracy** travels to a star 3 lightyears away from Earth on a starship with a $v = 0.6c$ speed, her outgoing journey to the star will take 5 years to complete, and her two-way trip back to Earth will take a total of 10 years. While Tracy's *twin* **Stacy**, who waits on Earth for her sister to return, ages 10 years during Tracy's voyage, Tracy will have aged only

$$t' = \sqrt{1 - \frac{v^2}{c^2}} 10\,\text{yrs} = \sqrt{1 - 0.6^2} 10\,\text{yrs} = 8\,\text{yrs} \tag{23}$$

during her voyage *because of time dilation* (or else, from Tracy's perspective, *because of a Lorentz contracted distance* of 2.4 lightyears to the star as seen from Tracy's reference frame).

This "time travel" of Tracy, a 2-year jump into Stacy's future (rather than the other way around — and this is the essence of so-called Twin Paradox[4]), occurs *because* Tracy takes the trouble of "jumping reference frames" unlike her "passive" twin Stacy who stays put in one and single reference frame. But because Stacy remains and lives in an inertial[5] reference frame *her* prediction of Tracy's aging given by (23) is valid, as opposed to "frame jumping" Tracy's use of

$$t' = \sqrt{1 - \frac{v^2}{c^2}} 8\,\text{yrs} = \sqrt{1 - 0.6^2} 8\,\text{yrs} = 6.4\,\text{yrs} \tag{24}$$

to predict how much Stacy's would age during her travels — when you are a "time machine" user like Tracy, you can't use the usual time dilation formula (21) with your own t inputs to deduce meaningful t' results! Logical, ain't it!?[6]

A detailed description of how time passes for Stacy and Tracy based on a clever usage of relativistic Doppler formula can be read in Hewitt73. Similar ideas are presented in a more compact form in Figure 2 (see caption for a detailed explanation).

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[4]Experimentally verified by comparing the readings of identical and synchronized precision clocks one of which is taken on a long and fast trip around the globe many many times.

[5]All relativistic results being derived here apply to non-accelerated reference frames only! Relativistic complications due to acceleration processes were treated separately by Albert Einstein in his work on "general relativity" first published in 1916.

[6]Here is a far out example of time-dilation/Lorentz-contraction: Consider a galaxy 10^6 light-years away from Earth (a typical intergalactic distance - Andromeda is $\sim$2.5 million light-years away), meaning that it takes light a million years to reach that galaxy. Next consider a very fast moving spaceship zipping by Earth with a speed v sufficiently close to c so that 10^6 light-years distance to the galaxy Lorentz contracts to merely 10 light-years in the reference frame of the spaceship — this requires $\frac{v}{c} \approx 1 - \frac{1}{2}10^{-10}$!!! Captain Tina, the pilot of the spaceship, will see in that case the *same galaxy* only 10 light-years away and approaching her at an incredible speed of $c(1 - \frac{1}{2}10^{-10}) \approx c$, and therefore reaching her in just about 10 years (while about a million years passes by on Earth)! That's the trick for traveling over a million light-year distance in merely ten years — welcome to Intergalactic Express ;-)

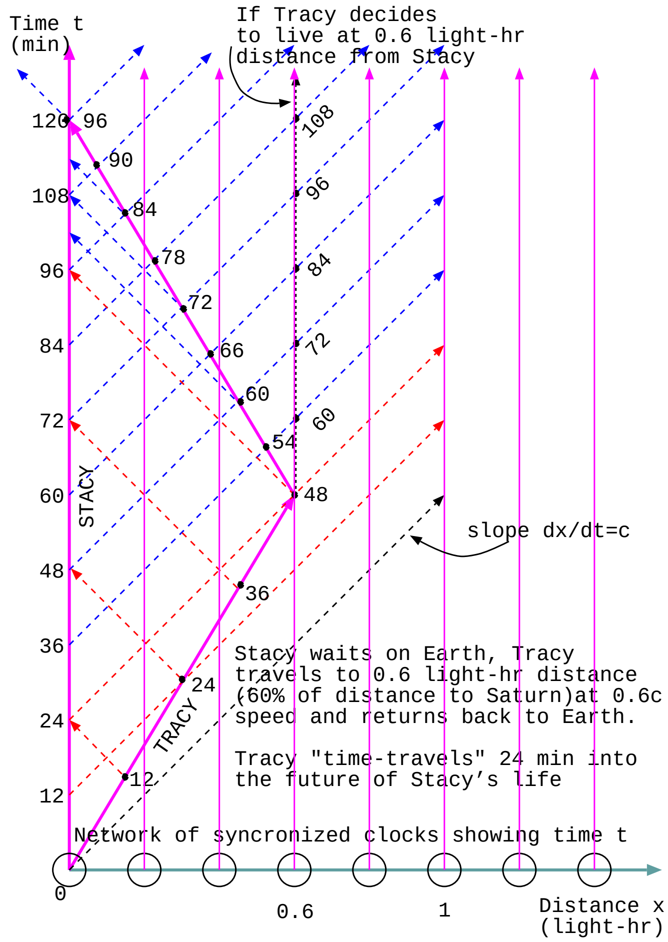


Figure 2: "Worldlines" of Stacy, Tracy, and clocks stationary in Stacy's reference frame are shown in magenta — the term worldline refers to trajectory curves of objects drawn in a selected reference frame. The worldlines of wave crests emitted by Stacy and Tracy's identical 12 min period oscillators are also shown in red and blue (dashed) according to the Doppler shifts they suffer before being observed. The 12 min period emissions are first redshifted to 24 min period during the outbound trip of Tracy to a distance of 0.6 light-hr at a 0.6c speed, and subsequently blueshifted to 6 min period during Tracy's inbound trip at the same speed.

Stacy emits 10 wave crests 12 min apart, *for Tracy to detect* 2 crests 24 min apart + 8 crests 6 min apart during her 96 min long voyage.
Tracy emits 8 wave crests 12 min apart, *for Stacy to detect* 4 crests 24 min apart + 4 crests 6 min apart during her 120 min long wait (for Tracy to return).
Stacy ages 120 min to emit 10 crests while Tracy ages only 96 min to emit only 8 crests in order to "time travel" into Stacy's future by 24 minutes.

Notice, we deduced the time-dilated travel time of Tracy (48 min for each leg) without using the time dilation formula (21) — keeping track of the Doppler shifted wave periods of Stacy's emissions was sufficient to come up with the correct aging times of both Stacy and Tracy! This can be regarded as an *independent confirmation* of phase invariance between reference frames, the root cause of time dilation (21).

1.8 Lorentz-Fitzgerald transformations

Time-dilation, Lorentz contraction, existence of a universal speed limit, twin paradox — these relativistic phenomena of the world that we live in are at odds with Galilean transformations (3-4) central to Newtonian mechanics. Today we know the Galilean transformations and Newtonian mechanics to be low-velocity approximations of a more fundamental paradigm of relativistic mechanics based on Lorentz-Fitzgerald transformations relating (t', x') to (t, x).

Galilean transformations $x' = x - vt$ and $t' = t$ are asymmetric in that while x' depends on both x and t, t' is only dependent on t. These asymmetric equations are clearly incapable of accounting for, say, time dilation. A generalized set of transformation equations where space-time coordinates (t', x') are linearly related to coordinates (t, x) can be written in the form

$$x' = \alpha x + \beta t \tag{25}$$

$$t' = \gamma t + \delta x \tag{26}$$

where coefficients α, β, γ, δ are to be adjusted so that the equations are compatible with time-dilation (21) or Lorentz contraction (22), as well as the relative velocity formula (18). With that procedure one obtains the Lorentz-Fitzgerald transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{27}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{28}$$

as verified next.

Suggested exercises:

1 Using the differential forms of (25-26), namely $dx' = \alpha dx + \beta dt$ and $dt' = \gamma dt + \delta dx$, within the relativistic velocity equation (18) expressed in terms of speeds $u' \equiv dx'/dt'$ and $u \equiv dx/dt$, show that $\alpha = \gamma$, $\beta = -v\gamma$, and $\delta = -\frac{v}{c^2}\gamma$, so that (25-26) can be re-written as

$$x' = \gamma(x - vt) \tag{29}$$

$$t' = \gamma(t - \frac{v}{c^2}x). \tag{30}$$

2 Using (29), determine in the moving frame and at $t' = 0$ the two endpoints x' of a stationary ruler extending from the origin to $x = L$ in the lab frame and show that the difference of the endpoints of the ruler in the moving frame is given by the Lorentz contraction formula (22) provided that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{31}$$

Equations (29-30) accompanied by (31) are in effect the Lorentz-Fitzgerald transformations (27-28). **Hint:** substitute for t in (29) the value deduced from (30) for $t' = 0$.

3 Using (28), determine the primed frame time t' for a clock at $x' = 0$ and show that the result is compliant with time-dilation (21).

4 Write down the transformation equations for (t, x) in terms of (t', x') derived from (27-28). **Hint:** swap x' with x and t' with t in (27-28) after replacing all v with $-v$; this should work since the motions of the two frames are relative but in opposite directions! You may still want to verify these the long way (algebraically) to "once and for all" clear all doubts you may have about your future uses of this neat trick.

5 Two lightbulbs located at $x = 0$ and $x = L$ in the lab frame "light up" simultaneously at time $t = 0$. (a) Calculate the coordinates x' of the lightbulbs at time $t' = 0$ in the primed frame. (b) Calculate the times t' when the lightbulbs light up in the primed frame — notice that they do not light up simultaneously as they do in the lab frame!

6 Re-derive the one-way relativistic Doppler formula expressed as $\omega' = \gamma(\omega - kv)$ by equating the phases of waveforms $\cos(\omega t - kx)$ and $\cos(\omega' t' - k' x')$ in lab and primed frames, using the result of (4) expressed as $x = \gamma(x' + vt')$ and $t = \gamma(t' + vx'/c^{2)}$ in terms of $\gamma = 1/\sqrt{1 - v^2/c^2}$. Note that in material media with refractive index $n = n(\omega)$, the relativistic Doppler shift formula becomes $\omega' = \omega\gamma(1 - n(\omega)v/c)$ after using $k = \frac{\omega}{c}n$. In material media the wave propagation velocities $v_p = \frac{c}{n(\omega)}$ and $v_p' = \frac{c}{n(\omega')}$ will be different in general, unlike in vacuum.

1.9 Summary and conclusions

We have presented here an unconventional introduction to **special relativity**.

Our treatment ends with, rather than starting with, Lorentz-Fitzgerald transformations. In our approach the invariance of c in all reference frames is not a postulate but a natural outcome of what we already knew from our study of TEM wave solutions of Maxwell's equations in ECE 350 (assuming that all equations of physics have the same form in all reference frames in relative uniform motion).

The absence of an absolute reference frame is also taken on naturally — and hence the notion that all speeds are relative — as the notion was already inherent in Galilean transformations.

Student should note that given (27-28), it follows that

$$c^2t^2 - x^2 = c^2t'^2 - x'^2,$$

that is, $c^2t^2 - x^2$ is a *relativistic invariant* unlike the space-time coordinates (x, t). Student should read about the implications of the invariance of $c^2t^2 - x^2$, known as the special relativistic *interval,* to learn more deeply about relativity and relativistic dynamics (relativistic versions of energy and momentum concepts). It may still be a good idea to take PHYS 225 ;-)

2 Lorentz invariance — relativistic dynamics

We are familiar with 3D vectors such as $\vec{r} = (x, y, z)$ and $\vec{k} = (k_x, k_y, k_z)$ and their dot products denoted and computed as $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$.

It is useful to extend such 3D vectors into so-called "4-vectors[7]" with four scalar components such as ct, x, y, and z having the same dimension and belonging to the lab frame which transform into the primed reference frames according to Lorentz-Fitzgerald transformations

$$ct' = \gamma(ct - \frac{v}{c}x), \;\; x' = \gamma(x - \frac{v}{c}ct), \;\; y' = y, \;\; z' = z. \tag{32}$$

These 4-vectors are denoted as

$$\mathbf{r} = (ct, x, y, z) = (ct, \vec{r}) \tag{33}$$

and dot products of a pair of 4-vectors $\mathbf{r}_1 = (ct_1, \vec{r}_1)$ and $\mathbf{r}_2 = (ct_2, \vec{r}_2)$ are denoted and computed as in

$$\mathbf{r}_1\mathbf{r}_2 = (ct_1, \vec{r}_1)\,(ct_2, \vec{r}_2) = c^2 t_1 t_2 - \vec{r}_1 \cdot \vec{r}_2 \tag{34}$$

in which the unexpected minus sign is not a typo and *intended* as an important feature of 4-vector algebra. With this definition of dot product for 4-vectors, we notice that

$$\mathbf{rr} = (ct, \vec{r})\,(ct, \vec{r}) = c^2 t^2 - \vec{r} \cdot \vec{r} \tag{35}$$

which is found to be identical with the 4-vector dot product $\mathbf{r}'\mathbf{r}'$ of another 4-vector

$$\mathbf{r}' = (ct', x', y', z') = (ct', \vec{r}\,') \tag{36}$$

with components ct', x', y', and z' obtained from ct, x, y, and z using the Lorentz-Fitzgerald transformations (32).

This fact, namely, $\mathbf{rr} = \mathbf{r}'\mathbf{r}'$, which can also be expressed as $\mathbf{r}^2 = \mathbf{r}'^2$, implies that the squared 2-norm $\mathbf{r}^2$ of 4-vector $\mathbf{r}$ is a *relativistic invariant,* meaning that it is unchanged when $\mathbf{r}$ is transformed into $\mathbf{r}'$ of the primed frame according to (32).

The same idea can in fact be extended to 2-norms of all 4-vectors as well as to dot products of any two 4-vectors that can be put together in relativistic calculations *so long as* 4-vector components transform like (32) — all such dot products are *relativistic invariants!*

Here is another 4-vector

$$\mathbf{k} = (\frac{\omega}{c}, \hat{x}k) \tag{37}$$

with components $\frac{\omega}{c}$, $k_x = k$, $k_y = k_z = 0$, which we know how to transform to the primed frame by using $k' = \frac{\omega'}{c}$ and the one-way Doppler formula for ω', resulting in

$$\omega' = \gamma(\omega - kv) \;\text{ and }\; k' = \gamma(k - \frac{v}{c^2}\omega), \tag{38}$$

[7] invented by Hermann Minkowski, Einstein's math prof at Zurich Tech, after he read Einstein's 1905 relativity paper written a few years after Einstein's graduation

indicating that the components of $\mathbf{k}$ transform like (32) with $ct \rightarrow \frac{\omega}{c}$ and $x \rightarrow k_x = k$ replacements, primed as well as unprimed, and with the transformed vector

$$\mathbf{k}' = (\frac{\omega'}{c}, \hat{x}k'). \tag{39}$$

Let's compare the inner products $\mathbf{kr}$ and $\mathbf{k}'\mathbf{r}'$ to see whether $\mathbf{kr}$ is a relativistic invariant as claimed. Here we go:

$$\mathbf{kr} = (\frac{\omega}{c}, \hat{x}k)\,(ct, \vec{r}) = \omega t - kx, \tag{40}$$

representing the phase of a wave propagating in the lab frame in same direction as the primed reference frame, whereas

$$\mathbf{k}'\mathbf{r}' = (\frac{\omega'}{c}, \hat{x}k')\,(ct', \vec{r}') = \omega' t' - k' x' = \gamma(\omega - kv)\gamma(t - \frac{v}{c^2}x) - \gamma(k - \frac{v}{c^2}\omega)\gamma(x - vt) = \omega t - kx \tag{41}$$

also. Yes, the wave phase $\mathbf{kr}$ is another relativistic invariant, just like $\mathbf{kk} = \mathbf{k^2}$.

Clearly the 4-vector $\mathbf{k}$ is a generalization of the wavevector $\vec{k}$ of an EM wave (a lá Minkowski), but what would the relativistic invariant $\mathbf{k^2}$ represent in physical terms? Before we answer that, consider the product

$$\hbar\mathbf{k} = (\frac{\hbar\omega}{c}, \hat{x}\hbar k) = (\frac{E}{c}, \hat{x}p) \equiv \mathbf{p}, \tag{42}$$

where we used $E = \hbar\omega$ and $p = \hbar k$ to denote the energy and momentum of a photon using our PHYS 214 knowledge. We can call $\mathbf{p}$ as the momentum-4-vector of a photon whose first element relates to the photon energy. Clearly photon energy and momentum E and p transform like $\omega/\hbar$ and $k/\hbar$, as in

$$E' = \gamma(E - pv) \text{ and } p' = \gamma(p - \frac{v}{c^2}E). \tag{43}$$

So, the relativistic invariants $\mathbf{p^2} \propto \mathbf{k^2} = \frac{\omega^2}{c^2} - k^2 = 0$ for the photon, since $k = \frac{\omega}{c}$. Also, since $\mathbf{p^2} = \frac{E^2}{c^2} - p^2 = 0$, the photon momentum $p = \frac{E}{c} = \frac{\hbar\omega}{c} = \hbar k$ is non-zero — despite the photon being massless — so long as $k = \frac{\omega}{c}$ is finite.

The momentum-4-vector $\mathbf{p} = (\frac{E}{c}, \vec{p})$ can be used for all particles, but with $\mathbf{p}^2 = \frac{E^2}{c^2} - p^2$ vanishing *only* for massless particles such as *photons* and *gravitons,* and being in general some non-zero $\frac{E_o^2}{c^2}$, where E_o denotes the particle energy in its rest frame, where, *by definition,* the particle momentum must be zero[8]. Thus

$$\frac{E^2}{c^2} - p^2 = \frac{E_o^2}{c^2} \;\Rightarrow\; E = \sqrt{E_o^2 + p^2c^2}. \tag{44}$$

Applying this for a slow moving particle with a non-zero mass m and *approximating* momentum p in the lab frame by "non-relativistic" momentum mv, we obtain

$$E \approx \sqrt{E_o^2 + m^2v^2c^2} = E_o\sqrt{1 + \frac{m^2v^2c^2}{E_o^2}} \approx E_o + \frac{1}{2}\frac{m^2v^2c^2}{E_o} = E_o + \frac{1}{2}mv^2 \tag{45}$$

[8]Note that there is no rest frame for a photon and thus no corresponding E_o.

after taking $E_o = mc^2 \gg mvc$ — this result, indicating that the energy E of slow moving particles can be interpreted as a sum of the particle rest energy E_o and a kinetic energy $T \approx \frac{1}{2}mv^2$, also establishes the fact that

$$E_o = mc^2 \quad (46)$$

in order to maintain the consistency of relativistic and non-relativistic particle dynamics.

To determine the *relativistic momentum* p of a fast moving particle in the lab frame, we make use of (43) with $p' = 0$ and $E' = E_o = mc^2$ for a particle in its own frame, and find out that $E = \gamma E_o$ (try showing this with a few steps of algebra) and

$$p = \frac{vE}{c^2} = \frac{v\gamma E_o}{c^2} = \frac{v\gamma mc^2}{c^2} = \gamma mv. \quad (47)$$

Using the relativistic p (47) within $E = \sqrt{m^2c^4 + p^2c^2} \equiv mc^2 + T$, we obtain the *relativistic kinetic energy* as

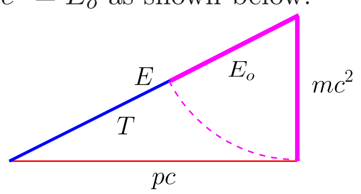
$$T = \sqrt{m^2c^4 + p^2c^2} - mc^2 = mc^2(\gamma - 1) \quad (48)$$

after some algebra. Consequently

$$E = mc^2 + T = mc^2 + mc^2(\gamma - 1) = \gamma mc^2. \quad (49)$$

In summary, all particles with mass m have energies $E = \gamma mc^2$, always a sum of a rest energy mc^2 and a kinetic energy T, the latter being $T \approx \frac{1}{2}mv^2$ in the non-relativistic regime when $|v| \ll c$.

Combination of (45) and (46) leads to $E^2 = p^2c^2 + m^2c^4$ that has the form of Pythagorean theorem for a right angle triangle with a base of length pc and height $mc^2 = E_o$ as shown below:



Even a non-moving particle has some energy mc^2 that can be tapped into in a nuclear reaction!

Suggested exercises:

1 Make use of (43) with $p' = 0$ and $E' = E_o = mc^2$ for a particle with mass m in its own frame to show that $E = \gamma E_o$.

2 Confirm (48) after substituting γmv for p within $\sqrt{m^2c^4 + p^2c^2} - mc^2$.

3 Using $E = \gamma E_o = \gamma mc^2$ and $p = \gamma mv$, the momentum-4-vector can also be expressed as

$$\mathbf{p} = (\frac{E}{c}, \hat{x}p) = (\gamma mc, \hat{x}\gamma mv) \equiv \gamma m\mathbf{v},$$

where $\mathbf{v} = (c, \hat{x}v)$ is the velocity-4-vector of a particle related to its position-4-vector $\mathbf{r} = (ct, \hat{x}x)$ via $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. Verify $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ from the differential form of (32) by showing that $\frac{dx}{dt} = v$ with $dx' = 0$, so that the primed frame is the rest frame of the particle. **Hint:** make use of the differential form $d\mathbf{r} = (cdt, \hat{x}dx)$ of $\mathbf{r} = (ct, \hat{x}x)$.

4 Show that $\mathbf{v}^2$ is not a relativistic invariant (but $\mathbf{p}^2$ is).

5 Note that in older texts on special relativity you will see the notation $E_o = m_o c^2$ for the energy of a particle in its rest frame and $E = mc^2$ in other frames, with $m \equiv \gamma m_o$ referred to as *relativistic mass* and using the term *rest mass* to refer to m_o — mass was seen as a parameter that increased with the particle velocity. In modern usage $E = \gamma mc^2$ with m being referred to as *invariant mass,* an intrinsic particle property which happens to be a relativistic invariant just like the particle charge q! In high energy particle physics books tables of m and q are provided for different types of particles such as electrons, muons, neutrinos, quarks, etc., all treated as relativistic invariants.