ECE 330: Power Circuits and Electromechanics

Homework 6 Solution

1. Problem 4.1

c. From homework 5, we note that \( \lambda(i, x) = \mu_0 W^2 N^2 \left( \frac{1}{g} + \frac{1}{2x} \right) i \), thus \( i(\lambda, x) = \frac{2gx\lambda(i,x)}{\mu_0 W^2 N^2 (g + 2x)} \).

We can compute the energy by:

\[
W_m(\lambda, x) = \int_0^\lambda i(\hat{\lambda}, x) d\hat{\lambda} = \int_0^\lambda \frac{2gx\hat{\lambda}}{\mu_0 W^2 N^2 (g + 2x)} d\hat{\lambda} = \frac{gx\lambda^2}{\mu_0 W^2 N^2 (g + 2x)}
\]

As such, the electric force using energy function is determined by:

\[
f_e(\lambda, x) = -\frac{\partial W_m(\lambda, x)}{\partial x} = -g\lambda^2 \frac{g + 2x - 2x}{\mu_0 W^2 N^2 (g + 2x)^2} = -\frac{g^2 \lambda^2}{\mu_0 W^2 N^2 (g + 2x)^2}
\]

d. We can compute the co-energy by:

\[
W'_m(i, x) = \int_0^i \lambda(\hat{i}, x) d\hat{i} = \int_0^i \mu_0 W^2 N^2 \left( \frac{g + 2x}{2gx} \right) \hat{i} d\hat{i} = \mu_0 W^2 N^2 \left( \frac{g + 2x}{2gx} \right) \frac{i^2}{2}
\]

As such, the electric force using energy function is:

\[
f_e(i, x) = \frac{\partial W'_m(i, x)}{\partial x} = \mu_0 W^2 N^2 i^2 \frac{d}{dx} \left( \frac{g + 2x}{2gx} \right) = -\mu_0 W^2 N^2 i^2 \frac{1}{4x^2}
\]

Substituting \( \lambda(i, x) \) into \( f^e(\lambda, x) \):

\[
f^e(\lambda, x) = -\frac{g^2}{\mu_0 W^2 N^2 (g + 2x)^2} \lambda^2
\]

\[
= -\frac{g^2}{\mu_0 W^2 N^2 (g + 2x)^2} \left( \mu_0^2 W^4 N^4 i^2 \left( \frac{2x + g}{2xg} \right)^2 \right)
\]

\[
= -\mu_0 W^2 N^2 i^2 \frac{g^2}{4x^2 g^2}
\]

\[
= -\mu_0 W^2 N^2 i^2 \frac{1}{4x^2} = f_e(i, x)
\]
2. Problem 4.2

b. Given the inductance \( L(\theta) \) and the current \( i \), the flux linkage for the coil is \( \lambda(i, \theta) = L(\theta)i = (L_1 + L_2 \cos(2\theta))i \). Therefore, we can compute the co-energy function by:

\[
W'_m(i, \theta) = \int_0^i \lambda(i, \theta)\hat{i}d\hat{i} = \int_0^\lambda (L_1 + L_2 \cos(2\theta))\hat{i}d\hat{i} = \frac{1}{2}(L_1 + L_2 \cos(2\theta))i^2
\]

As such, the electric force using co-energy function is determined by:

\[
f^e = \frac{\partial W_m(i, x)}{\partial x} = -L_2 \sin(2\theta)i^2
\]
For the given electromechanical system, the reluctance is \( R = \frac{x}{\mu_0 A} = \frac{x}{\mu_0 W^2} \) with the flux linkage to be

\[
\lambda(i, x) = N\phi = N^2 i \frac{x}{R} = \frac{N^2 i \mu_0 W^2}{x}
\]

Therefore, we can compute the co-energy function by:

\[
W'_m(i, x) = \int_0^i \lambda(\hat{i}, x) d\hat{i} = \frac{N^2 i^2 \mu_0 W^2}{2x}
\]

As such, the electric force using co-energy function is determined by:

\[
f^e = \frac{\partial W'_m(i, x)}{\partial x} = -\frac{N^2 i^2 \mu_0 W^2}{2x^2}
\]

Correspondingly, the current is expressed in terms of the flux linkage:

\[
i(\lambda, x) = \frac{\lambda x}{N^2 \mu_0 W^2}
\]

We can compute the energy by:

\[
W_m(\lambda, x) = \int_0^\lambda i(\hat{\lambda}, x) d\hat{\lambda} = \frac{\lambda^2 x}{2N^2 \mu_0 W^2}
\]
4. Problem 4.6

For the given electromechanical system, the reluctances are \( R_x = \frac{x}{\mu_0 A} \) and \( R_g = \frac{g}{\mu_0 A} \)

![Figure 1: Problem 4.6](image)

with:

\[
Ni = \phi_1 R_g + (\phi_1 + \phi_2) R_x
\]
\[
Ni = \phi_2 R_g + (\phi_1 + \phi_2) R_x
\]

As such, \( \phi_1 = \phi_2 = \frac{Ni}{2R_x + R_g} \) and the flux linkage \( \lambda(i, x) = N\phi_1 + N\phi_2 = \frac{2N^2i}{2R_x + R_g} = \frac{2\mu_0 AN^2i}{2x + g} \). Therefore, with the flux we can compute the co-energy function by:

\[
W'_m(i, x) = \int_0^i \lambda(\hat{i}, x) d\hat{i} = \frac{\mu_0 AN^2i^2}{2x + g}
\]

As such, the electric force using co-energy function is determined by:

\[
f^e = \frac{\partial W'_m(i, x)}{\partial x} = -\frac{2\mu_0 AN^2i^2}{(2x + g)^2}
\]

Correspondingly, the current is expressed in terms of the flux linkage:

\[
i(\lambda, x) = \frac{2x + g}{2\mu_0 AN^2} \lambda
\]

We can compute the energy by:

\[
W_m(\lambda, x) = \int_0^\lambda i(\hat{\lambda}, x) d\hat{\lambda} = \frac{2x + g}{4\mu_0 AN^2} \lambda^2
\]
5. **Problem 4.10**
The co-energy of a device is $W'_m = \frac{i^3}{4x}$

(a) Find $\lambda$ as a function of $i$ and $x$.

$$\lambda = \frac{\partial W'_m}{\partial i} = \frac{3i^2}{4x}$$

(b) Find the force of electrical origin.

$$f_e = \frac{\partial W'_m}{\partial x} = -\frac{i^3}{4x^2}$$

(c) Find the energy stored in the coupling field $W_m$.

$$W_m = \lambda i - W'_m = \frac{3i^3}{4x} - \frac{i^3}{4x} = \frac{i^3}{2x}$$