ECE 330 (Spring 2018) | **Instructors:** S. Bose and H. Sowidan

Midterm 2

Duration: 90 minutes | Total points: 100

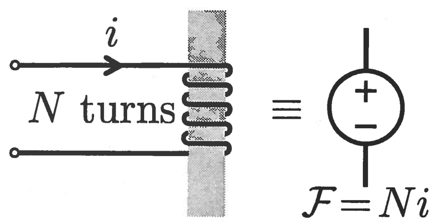
Name: ___Solution___________________________.

Section (Tick one): C (09:30 AM) ________ F (02:00 PM) ________.

Scores (For official use only):

Problem 1:__________/25, Problem 2:__________/25,
Problem 3:__________/25, Problem 4:__________/25. **Total score:**______________/100.

Relevant formulae



$\lambda = Li$(if linear) $\qquad v = \frac{d\lambda}{dt}$

$$W_m(\lambda, x) = \int_0^\lambda i(\hat{\lambda}, x)d\hat{\lambda} \qquad W'_m(i, x) = \int_0^i \lambda(\hat{i}, x)d\hat{i} \qquad \lambda = \frac{\partial W'_m(i,x)}{\partial i} \qquad i = \frac{\partial W_m(\lambda,x)}{\partial \lambda}$$

$$f^e(\lambda, x) = -\frac{\partial W_m(\lambda,x)}{\partial x} \qquad f^e(i, x) = \frac{\partial W'_m(i,x)}{\partial x} \qquad T^e(\lambda, \theta) = -\frac{\partial W_m(\lambda,\theta)}{\partial \theta} \qquad T^e(i, \theta) = \frac{\partial W'_m(i,\theta)}{\partial \theta}$$

$$W_m + W'_m = \lambda i \qquad EFE_{a\to b} = \int_a^b i d\lambda \qquad EFM_{a\to b} = -\int_a^b f^e dx \qquad EFM_{a\to b} = -\int_a^b T^e d\theta$$

$$EFE_{a\to b} + EFM_{a\to b} = W_m|_b - W_m|_a$$

For the state space form, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = [x_1, x_2, ..., x_n]^\top$ and $\mathbf{f} = [f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x})]^\top$, the linearized system at an equilibrium point, $\mathbf{x}^e$, is defined as:
$\Delta\dot{\mathbf{x}} = \nabla\mathbf{f}(\mathbf{x})|_{\mathbf{x}^e} \cdot \Delta\mathbf{x}$, where $\nabla\mathbf{f}$ is the Jacobian of $\mathbf{f}(\mathbf{x})$

For the state space form, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *Euler*'s method gives the state vector, $\mathbf{x}$, at time step, k, as:
$\mathbf{x}(t_k) \approx \mathbf{x}(t_{k-1}) + (t_k - t_{k-1}) \cdot \mathbf{f}\big(\mathbf{x}(t_{k-1})\big)$

$$\begin{array}{l}\dot{\mathbf{x}} = A\mathbf{x},\\ \text{eigs}(A) = \{\lambda_1, \ldots, \lambda_n\}\end{array} \implies \begin{cases}\text{Re}\{\lambda_i\} < 0\, \forall\, i \implies \text{stable}\\ \text{Re}\{\lambda_i\} > 0 \text{ for any } i \implies \text{unstable}\\ \text{Re}\{\lambda_i\} \le 0\, \forall\, i, \text{ and } \text{Re}\{\lambda_i\} = 0 \text{ for some } i \implies \text{marginally stable.}\end{cases}$$

Problem 1 [25 points]

The flux linkages in an energy-conservative electromechanical system are given by

$$\lambda_a = L_a i_a + (M\cos\theta) i_b,$$
$$\lambda_b = L_b i_b + (M\cos\theta) i_a + (M\sin\theta) i_c,$$
$$\lambda_c = L_c i_c + (M\sin\theta) i_b,$$

where L_a, L_b, L_c and M are positive constants, and i_a, i_b, i_c are currents into the system.

(a) Is the system electrically linear? [1 point]

(b) How many electrical and mechanical ports does the system have? [2 + 2 points]

(c) Find the co-energy $W'_m(i_a, i_b, i_c, \theta)$ for this system. [12 points]

(d) Compute the torque of electric origin $T_e(i_a, i_b, i_c, \theta)$. [3 points]

(e) Compute the maximum absolute value of $T_e(i_a = I, i_b = I, i_c = I, \theta)$ over $\theta \in [0, 2\pi]$, where I is a positive constant. Also, report *all* values of $\theta \in [0, 2\pi]$, where this maximum is attained. [5 points]

A. Linear. λ's are linear functions of $i_a, i_b, + i_c$.

B. Electrical Ports: $\left.\begin{matrix} i_a \\ i_b \\ i_c \end{matrix}\right\} 3$

Mechanical Ports: $\theta \} 1$

C.
$$W_m'(i_a, i_b, i_c, \theta) = \int_0^{i_a} \lambda_a(i_a', 0, 0, \theta)\, di_a' + \int_0^{i_b} \lambda_b(i_a, i_b', 0, \theta)\, di_b'$$
$$+ \int_0^{i_c} \lambda_c(i_a, i_b, i_c', \theta)\, di_c'$$
$$= \int_0^{i_a} L_a i_a'\, di_a' + \int_0^{i_b} M\cos(\theta) i_a + L_b i_b'\, di_b'$$
$$+ \int_0^{i_c} L_c i_c' + M\sin(\theta) i_b\, di_c'$$

$$\boxed{W_m'(i_a, i_b, i_c, \theta) = \frac{1}{2} L_a i_a^2 + \frac{1}{2} L_b i_b^2 + M\cos(\theta) i_a i_b + \frac{1}{2} L_c i_c^2 + M\sin(\theta) i_b i_c}$$

D. $T_e(i_a, i_b, i_c, \theta) = \dfrac{\partial W'(i_a, i_b, i_c, \theta)}{\partial \theta}$

$$= -M\sin(\theta)\, i_a i_b + M\cos(\theta)\, i_b i_c$$

E. Extrema (maximum/minimum):

$$\frac{\partial T_e}{\partial \theta} = 0 \rightarrow -M\cos(\theta) I^2 - M\sin(\theta) I^2 = 0$$

$$\cos(\theta) + \sin(\theta) = 0$$

$$\tan(\theta) = -1$$

$$\boxed{\theta_{ex} = \tan^{-1}(-1) = \frac{3\pi}{4} + n\pi, \text{ for all } n \in \mathbb{Z}}$$

$$T_e(\theta_{ex}) = -M\sin(\theta_{ex}) I^2 + M\cos(\theta_{ex}) I^2 \xrightarrow{\theta_{ex} = \frac{3\pi}{4}} = -MI^2\sqrt{2} \leftarrow \text{minimum}$$

$$\xrightarrow{\theta_{ex} = \frac{7\pi}{4}} = +MI^2\sqrt{2} \leftarrow \text{maximum}$$

$$\boxed{|T_e|_{max} = MI^2\sqrt{2}}$$

Problem 2 [25 points]

The flux linkage in an energy-conservative electromechanical system is given by

$$\lambda(i, x) = \gamma\,(x - x_0)^2\, i,$$

where i denotes the current into the electrical subsystem and x defines the geometry of the mechanical subsystem. Also, $x_0 = 1$ m, and $\gamma = 2$ H/m^2. The system is being operated on the closed cycle $a \to b \to c \to d \to a$ as indicated in Figure 1.

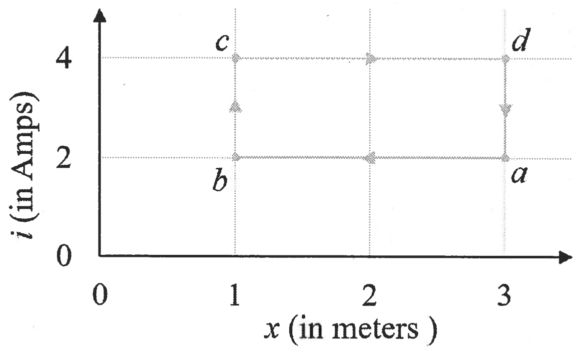


Figure 1: Operating cycle of the electromechanical system.

(a) Compute the co-energy $W'_m(i, x)$ in terms of γ and x_0. [4 points]

(b) Compute the force of electrical origin $f_e(i, x)$ in terms of γ and x_0. [3 points]

(c) Compute the "energy from mechanical" for each path (e.g., $a \to b$) of the cycle shown in Figure 1 (e.g., $\mathrm{EFM}|_{a\to b}$). Then compute the "energy from mechanical" for the full cycle (i.e., $\mathrm{EFM}|_{\text{cycle}}$). Fill in Table 1 with your results. [15 points]

Path	EFM
$a \to b$	16J
$b \to c$	0
$c \to d$	-64J
$d \to a$	0
cycle	- 48J

Table 1: EFM for each path and the complete cycle

(d) Based on your answer in part (c), state whether the electromechanical system is operating as a motor or a generator over the cycle. [1 points]

(e) How would your answers in part (c) & (d) change if the direction of the cycle as indicated by Figure 1 were reversed, i.e., it is operated over the cycle $a \to d \to c \to b \to a$? [1 + 1 points]

A. $W'_m(i,x) = \int_0^{i} \lambda(i',x)\, di' = \int_0^{i} \gamma (x-x_0)^2 i'^2\, di'$

$= \boxed{\frac{1}{2}\gamma (x-x_0)^2 i^2}$

B. $f^e(i,x) = \dfrac{\partial W_m'(i,x)}{\partial x} = 2\left[\frac{1}{2}\gamma (x-x_0)\, i^2\right]$

$= \boxed{\gamma (x-x_0)\, i^2}$

C. $EFM\Big|_{a\to b} = -\int_{x=3}^{1} f^e(i=2,x)\, dx = -\int_3^1 (2)(x-1)(2)^2 dx = -8\cdot\frac{1}{2}(x-1)^2\Big|_{x=3}^{1}$

$= -4\cdot\left[\cancelto{0}{(1-1)^2} - (3-1)^2\right] = \boxed{+16\ J}$

$EFM\Big|_{b\to c} = -\int_{x=1}^{1} \cancelto{0}{f^e(i,x)}\, dx = \boxed{0\ J}$

$EFM\Big|_{c\to d} = -\int_{x=1}^{3} f^e(i=4,x)\, dx = -\int_1^3 (2)(x-1)(4)^2 dx = -32\cdot\frac{1}{2}(x-1)^2\Big|_{x=1}^{3}$

$= -16\cdot\left[(3-1)^2 - \cancelto{0}{(1-1)^2}\right] = \boxed{-64\ J}$

$EFM\Big|_{d\to a} = -\int_{x=3}^{3} \cancelto{0}{f^e(i,x)}\, dx = \boxed{0\ J}$

$EFM\Big|_{cycle} = EFM\Big|_{a\to b} + EFM\Big|_{b\to c} + EFM\Big|_{c\to d} + EFM\Big|_{d\to a}$

$= (+16) + 0 + (-64) + 0 = \boxed{-48\ J}$

D. $EFM\big|_{cycle} < 0$ $\therefore$ System acts as motor

E. Part (c) $\rightarrow$ All values of EFM would have opposite signs.

Part (d) $\rightarrow$ $EFM\big|_{cycle} > 0$ $\therefore$ system acts as generator

Problem 3 [25 points]

A translational electromechanical dynamical system is shown in Figure 2.

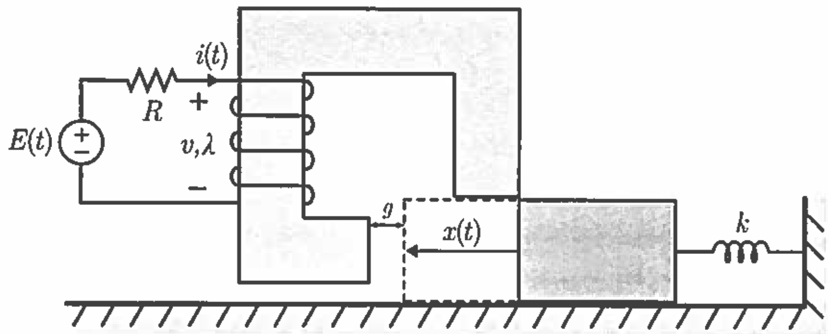


Figure 2: Electromechanical system.

In this system, a voltage source E in series with a resistor R induces a current i and flux linkage λ in the coil surrounding a looped magnetic structure with a lossless core. A metallic block with mass m attached to a spring with constant k is oriented in such a way as to create a variable gap g in the magnetic structure. The block slides along a frictionless surface and is only able to move in the lateral direction denoted by the position x. When $x = 0$, the spring is uncompressed.

The inductance of the magnetic structure is given by $L(x) = L_o x$. Additionally, note that $i(t), x(t)$, and $E(t)$ are all functions of time t.

(a) Derive the force of electric origin $f^e(i, x)$ acting on the block. (Hint: Derive $W'_m(i, x)$) [4 points]

(b) Derive the mechanical equation of the system as a function of the inputs i, x, and E and the parameters R, L_o, and k. Complete the equation below. [4 points]

$$m\ddot{x} = \underbrace{L_o i^2(t)/2}_{\text{force of electric origin}} - \underbrace{k x}_{\text{spring force}}$$

(c) Derive the electrical equation of the system as a function of the inputs i, x, and E and the parameters R, L_o, and k. Complete the equation below. [7 points]

$$E = \underbrace{i(t) R}_{\text{voltage drop across resistor}} + \underbrace{L i(t) \frac{dx}{dt} + L_o x(t) \frac{di}{dt}}_{\text{coil voltage}}$$

(d) Derive the dynamical description of the differential equations in part (b) & (c) in state space form. Use $X = [x(t), \dot{x}(t), i(t)]$ as your state vector. [8 points]

(e) If the magnetic core has hysteresis loss, can you still compute f^e from W'_m as you did in part (a)? Comment briefly. [2 point]

a) $W_m'(i,x) = \int_0^{i} \lambda(\hat{i},x)\,d\hat{i}$

now $\lambda(\hat{i},x) = L(x)\cdot \hat{i}(t)$

$= L_0 x(t)\cdot \hat{i}(t)$

$\therefore W_m'(i,x) = \int_0^{i} L_0 x(t)\,\hat{i}(t)\,d\hat{i} = \dfrac{L_0 x(t)\, i^2(t)}{2}$

$f^e(i,x) = \dfrac{dW_m'(i,x)}{dx} = \dfrac{L_0 i^2(t)}{2}$

b) $m\ddot{x} = f^e(i,x) - f_{spring}$

$m\ddot{x} = \dfrac{L_0 i^2(t)}{2} - Kx$

c) $E = i(t)R + V_A = i(t)R + \dfrac{d\lambda(t)}{dt}$

now $\dfrac{d\lambda(t)}{dt} = \dfrac{\partial\lambda}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\lambda}{\partial i}\dfrac{di}{dt}$

$= L\,i(t)\dfrac{dx}{dt} + L_0 x(t)\dfrac{di}{dt}$

$E = i(t)R + L\,i(t)\dfrac{dx}{dt} + L_0 x(t)\dfrac{di}{dt}$

d) $X_1 = x(t)$ $\quad\therefore \dot{X}_1 = \dot{x}(t) = X_2$ — ①

$X_2 = \dot{x}(t)$

$X_3 = i(t)$

from b)

$$m\dot{X}_2 = L_0 \frac{X_3^2}{2} - kX_1$$

$$\Rightarrow \dot{X}_2 = \frac{L_0}{2m} X_3^2 - kX_1 \quad \text{—— ②}$$

from c)

$$E - X_3R - LX_3X_2 = L_0 X_1 \dot{X}_3$$

$$\Rightarrow \dot{X}_3 = \frac{1}{L_0 X_1}\left[E - X_3R - LX_3X_2\right] \text{ — ③}$$

Putting ① ② & ③ together, we get the state space form.

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} X_2 \\ \frac{L_0}{2m} X_3^2 - kX_1 \\ \frac{1}{L_0X_1}\left[E - X_3R - LX_3X_2\right] \end{bmatrix}$$

e) If the magnetic core has hysteresis loss, the coupling field or medium becomes a non conservative one. Therefore, we cannot compute f^e and W_m as we did in part a) since the computions made in part a) are independent of path chosen which holds for conservative setting

Problem 4 [25 points]

The state space model of a dynamical system is described by

$$\dot{x}_1 = 2x_2,$$
$$\dot{x}_2 = -2x_2 + 4x_1 + x_1^2.$$

(a) What is the order of the dynamical system in the above state space representation? Is this a linear dynamical system? [1 + 1 point]

(b) Find all equilibrium points. [4 points]

(c) Linearize the state space model around each equilibrium point (you derived in part (b)). Your linearized systems should be given in state space form (i.e., $\dot{X} = AX$). [6 points]

(d) State whether each linearized system in part (c) is stable, unstable, or marginally stable. [4 points]

(e) Given the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ at $t = 0$, fill in the missing entries in Table 1 using Euler's method for numerical integration with a time step of $\Delta t = 0.1$. [7 points]

t	$x_1^{\text{Euler}}(t)$	$x_2^{\text{Euler}}(t)$
0	1	0
0.1	1	0.5
0.2	1.1	0.9
0.3	1.28	1.28

Table 2: Euler's method with $\Delta t = 0.1$

(f) Suppose you could calculate $x_1(t)$ and $x_2(t)$ exactly. How would you expect the absolute errors $e_1 = |x_1(t) - x_1^{\text{Euler}}(t)|$ and $e_2 = |x_2(t) - x_2^{\text{Euler}}(t)|$ to behave as $t \to \infty$? [1 + 1 points]

a) Order - 2 .. No it is not $\because$ $\dot{x}_2 = -2x_2 + 4x_1 + x_1^2$

Non Linear.

b) $2x_2 = 0 \Rightarrow x_2 = 0$

$-2x_2 + 4x_1 + x_1^2 = 4x_1 + x_1^2 = 0 \Rightarrow x_1(x+4) = 0$

eqbm points : $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

c) $\dot{X} = AX$ where $A = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}$.

here $f_1(x_1,x_2) = 2x_2$

$f_2(x_1,x_2) = -2x_2 + 4x_1 + x_1^2$

$$A = \begin{bmatrix} 0 & 2 \\ 4+2x_1 & -2 \end{bmatrix}$$

eqbm pt $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eqbm pt $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$$A_1 = \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}$$

$\dot{X} = A_1 X$

$$A_2 = \begin{bmatrix} 0 & 2 \\ -4 & -2 \end{bmatrix}$$

$\dot{X} = A_2 X$

d) for A_1, we compute eigenvalues and the we do it for A_2.

$$A_1 - \lambda I = \begin{bmatrix} -\lambda & 2 \\ 4 & -2-\lambda \end{bmatrix}$$

char $(A_1 - \lambda I) = 0$

$\Rightarrow \lambda(\lambda+2) - 8 = 0$

$\lambda^2 + 2\lambda - 8 = 0$

$\lambda^2 + 4\lambda - 2\lambda - 8 = 0$

$\lambda(\lambda+4) - 2(\lambda+4) = 0$

$\Rightarrow (\lambda-2)(\lambda+4) = 0$

$\lambda_1 = 2$ $\lambda_2 = -4$

$\rightarrow$ +ve eigen value and hence the equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable.

$$A_2 - \lambda I = \begin{bmatrix} -\lambda & 2 \\ -4 & -2-\lambda \end{bmatrix}$$

$\text{char}(A_2 - \lambda I) = 0$

$\lambda(\lambda+2) + 8 = 0$

$\lambda^2 + 2\lambda + 8 = 0$

$$\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{4-32}}{2}$$

$$= \frac{-2 \pm \sqrt{-28}}{2}$$

$$= \frac{-2 \pm 2\sqrt{7}\,i}{2}$$

$$= -1 \pm \sqrt{7}\,i$$

$\therefore$ $Re(\lambda_1, \lambda_2)$ is $-ve$, eqbm pt $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$ is stable.

e) $$\begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix} = \begin{bmatrix} x_1(t_{k-1}) \\ x_2(t_{k-1}) \end{bmatrix} + 0.1 \begin{bmatrix} f_1(x_1(t_{k-1}), x_2(t_{k-1})) \\ f_2(x_1(t_{k-1}), x_2(t_{k-1})) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0.1) \\ x_2(0.1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0.2) \\ x_2(0.2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + 0.1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0.3) \\ x_2(0.3) \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} + 0.1 \begin{bmatrix} 1.8 \\ 3.81 \end{bmatrix} = \begin{bmatrix} 1.28 \\ 1.281 \end{bmatrix}$$

f) The error would be very large(∞), since we started from an initial condition $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is a slight perturbation to an unstable equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore $x_1^{Euler}(t)$ & $x_2^{Euler}(t)$ as $t \to \infty$ diverges or blows up or becomes very large.