ECE330: Power Circuits & Electromechanics

Lecture 15. Nonlinear materials and multiple coils

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Schedule

- ~~Fri 3/13: Energy via line integrals~~
- **Mon 3/16: Spring break**
- **Wed 3/18: Spring break**
- **Fri 3/20: Spring break**
- ~~Mon 3/23: Co-energy via line integrals~~
- ~~**Wed 3/11: Homework 6 + Review**~~
- Fri 3/27: Nonlinear materials and multiple coils
- Mon 3/30: Conversion cycles

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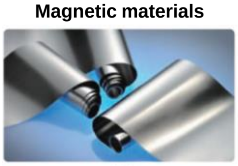
Magnetic materials



Multi-coil inductors

Voltage of mechanical origin
Force of electric origin
Energy, Co-Energy

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Today

- Nonlinear inductors
- Two-coil inductors
- Bonus: Inductance matrix

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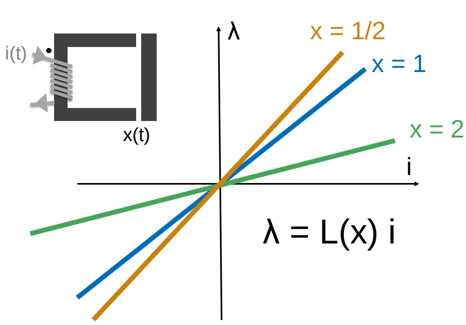
Electrically linear inductor



λ = L(x) i

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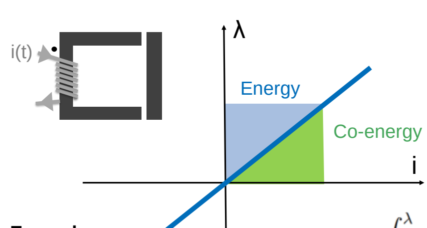
Electrically linear inductor



Example.

$$\text{Co-Energy} = \frac{\mu_0 A N^2}{2x} i^2$$

$$f^e = -\frac{\mu_0 A N^2}{2} \frac{i^2}{x^2}$$

$$\text{Energy} = \int_0^{\lambda} i(\lambda', x)\, \mathrm{d}\lambda'$$

$$\text{Co-energy} = \int_0^{i} \lambda(i', x)\, \mathrm{d}i'$$

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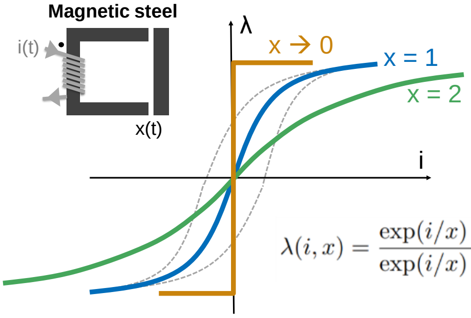
Realistic inductor



$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

How does saturation affect force?

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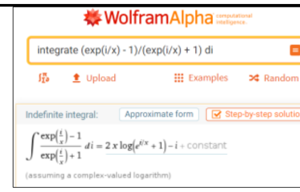
Computing forces

$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

$$\text{Energy} = \int_0^{\lambda} i(\lambda', x)\, d\lambda'$$



$$\text{Co-energy} = \int_0^{i} \lambda(i', x)\, di' = \int_0^{i} \frac{\exp(i'/x)-1}{\exp(i'/x)+1}\, di'$$

$$= 2x \log\left(\frac{\exp(i/x)+1}{2}\right) - i$$

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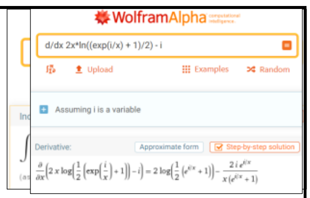
Computing forces

$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

$$\text{Energy} = \int_0^{\lambda} i(\lambda', x)\, d\lambda'$$



$$\text{Co-energy} = \int_0^{i} \lambda(i', x)\, di' = \int_0^{i} \frac{\exp(i'/x)-1}{\exp(i'/x)+1}\, di'$$

$$= 2x \log\left(\frac{\exp(i/x)+1}{2}\right) - i$$

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}] = \frac{-2i/x}{1+\exp(-i/x)} + 2\log\left(\frac{\exp(i/x)+1}{2}\right)$$

Imagine doing this with energy...

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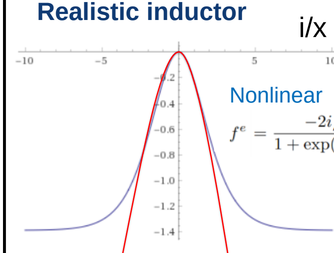
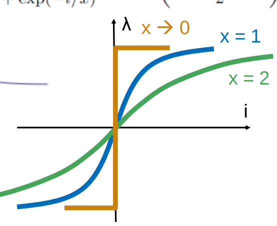
Realistic inductor



Finite force, saturation of stored energy

Nonlinear

$$f^e = \frac{-2i/x}{1+\exp(-i/x)} + 2\log\left(\frac{\exp(i/x)+1}{2}\right)$$



Linear

$$f^e = -\frac{\mu_0 A N^2}{2}\frac{i^2}{x^2}$$

Infinite force, infinite stored energy

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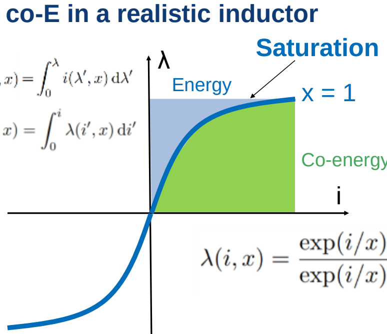
E and co-E in a realistic inductor

$$\text{Energy}(\lambda, x) = \int_0^{\lambda} i(\lambda', x)\, d\lambda'$$

$$\text{Co-energy}(i, x) = \int_0^{i} \lambda(i', x)\, di'$$



$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

In general, Energy ≠ Co-energy

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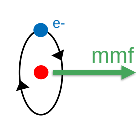
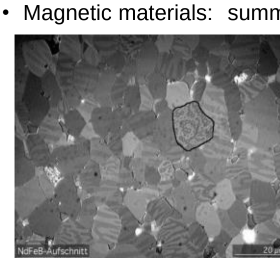
Aside: Magnetic materials

- Inherent atomic currents
- Most materials: cancellation
- Magnetic materials: summation





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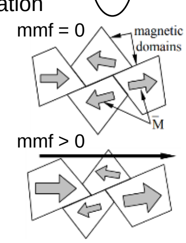

Saturation & Remanence

Once magnetic domains depleted, no more storage

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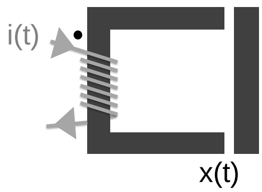
Concept question:

Where does the inductor store its energy?



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Today

- Nonlinear inductors
- Two-coil inductors
- Bonus: Inductance matrix

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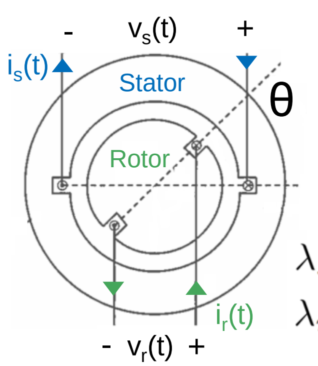

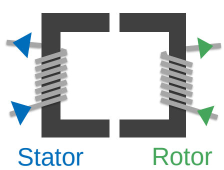

$$\lambda_s = L_s i_s + M i_r$$
$$\lambda_r = M i_s + L_r i_r$$

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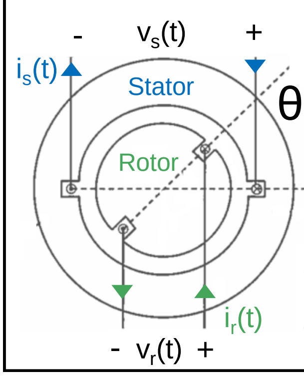

Which term(s) are θ-dependent?

A) Ls, M, Lr Reality

~~B) Ls, M~~

~~C) Lr, M~~

~~D) Ls, Lr~~

E) M Good approximation

Core geometry ~fixed wrt θ.

Coil position free wrt θ.

$$\lambda_s = L_s i_s + M i_r$$
$$\lambda_r = M i_s + L_r i_r$$

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Speed voltage for two coils

$$\lambda_s = L_s i_s + M(\theta) i_r$$
$$\lambda_r = M(\theta) i_s + L_r i_r$$

$$\boxed{M'(\theta) = \frac{\partial}{\partial \theta} M(\theta)}$$

$$v_s = L_s \frac{di_s}{dt} + M(\theta)\frac{di_r}{dt} \quad + M'(\theta) i_r \frac{d\theta}{dt}$$
$$v_r = \underbrace{M(\theta)\frac{di_s}{dt} + L_r \frac{di_r}{dt}}_{\text{Transformer}} \quad \underbrace{+ M'(\theta) i_s \frac{d\theta}{dt}}_{\text{Speed}}$$

Textbook
p. 114
Eqn. 4.35, 4.36

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Energy over two coils

$$\lambda_s = L_s i_s + M(\theta) i_r$$
$$\lambda_r = M(\theta) i_s + L_r i_r$$

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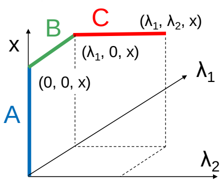
Integration path for two coils

$\underline{F} = (i_1, i_2, -f)$

$\underline{r} = (\lambda_1, \lambda_2, x)$



A: Move x into position with no flux

B: Charge coil 1 with fixed x and de-energized coil 2

C: Charge coil 2 with fixed x and energized coil 1

by def 0

$$\mathrm{Energy}(\lambda_1, \lambda_2, x) - \cancel{\mathrm{Energy}(0,0,0)}$$

$$= \cancel{\int_A \underline{F}\cdot \mathrm{d}\underline{r}} + \int_B \underline{F}\cdot \mathrm{d}\underline{r} + \int_C \underline{F}\cdot \mathrm{d}\underline{r}$$

0

$$\int_A \underline{F}\cdot \mathrm{d}\underline{r} = \cancel{\int_A -f\,\mathrm{d}x} = 0$$

0 no flux no force

$$\int_B \underline{F}\cdot \mathrm{d}\underline{r} = \int_B i_1(\lambda_1', \lambda_2', x')\,\mathrm{d}\lambda_1'$$

$$= \int_0^{\lambda_1} i_1(\underline{\lambda_1', 0, x})\,\mathrm{d}\lambda_1'$$

$$\int_C \underline{F}\cdot \mathrm{d}\underline{r} = \int_C i_2(\lambda_1', \lambda_2', x')\,\mathrm{d}\lambda_2'$$

$$= \int_0^{\lambda_2} i_2(\underline{\lambda_1, \lambda_2', x})\,\mathrm{d}\lambda_2'$$

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$$\mathrm{Energy}(\lambda_1, \lambda_2, \lambda_3 \ldots, x) = \int_0^{\lambda_1} i_1(\lambda_1', 0, 0, \ldots, x)\,\mathrm{d}\lambda_1'$$

always fixed

$$+ \int_0^{\lambda_2} i_2(\lambda_1, \lambda_2', 0, \ldots, x)\,\mathrm{d}\lambda_2'$$

$$+ \int_0^{\lambda_3} i_3(\lambda_1, \lambda_2, \lambda_3', \ldots, x)\,\mathrm{d}\lambda_3'$$

$$+ \cdots$$

"Charge one coil at a time"

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Co-Energy over two coils

$$\lambda_s = L_s i_s + M(\theta) i_r$$

$$\lambda_r = M(\theta) i_s + L_r i_r$$

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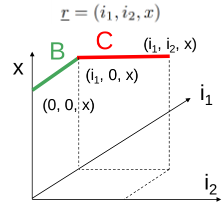
Co-energy and torque for two-coils

$\underline{F} = (\lambda_1, \lambda_2, f)$

$\underline{r} = (i_1, i_2, x)$

$\lambda_1 = L_1 i_1 + M(\theta) i_2$

$\lambda_2 = M(\theta) i_1 + L_2 i_2$



B: Charge coil 1 with fixed x and de-energized coil 2

C: Charge coil 2 with fixed x and energized coil 1

$$\mathrm{Co\text{-}energy}(i_1, i_2, \theta)$$

$$= \int_0^{i_1} \lambda_1(i_1', 0, \theta)\,\mathrm{d}i_1' + \int_0^{i_2} \lambda_2(i_1, i_2', \theta)\,\mathrm{d}i_2'$$

Charge coil 1 — Charge coil 2 with coil 1 charged

$$= \int_0^{i_1} [L_1 i_1' + M(\theta) 0]\,\mathrm{d}i_1'$$

$$+ \int_0^{i_2} [M(\theta) i_1 + L_2 i_2']\,\mathrm{d}i_2'$$

$$\boxed{= L_1 \frac{i_1^2}{2} + M(\theta) i_1 i_2 + L_2 \frac{i_2^2}{2}}$$

$$\tau_e = M'(\theta) i_1 i_2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

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$$\mathrm{Energy}(\lambda_1, \lambda_2, \lambda_3 \ldots, x) = \int_0^{\lambda_1} i_1(\lambda_1', 0, 0, \ldots, x)\,\mathrm{d}\lambda_1'$$

always fixed

$$+ \int_0^{\lambda_2} i_2(\lambda_1, \lambda_2', 0, \ldots, x)\,\mathrm{d}\lambda_2'$$

$$+ \int_0^{\lambda_3} i_3(\lambda_1, \lambda_2, \lambda_3', \ldots, x)\,\mathrm{d}\lambda_3'$$

$$+ \cdots$$

"Charge one coil at a time"

$$\mathrm{Co\text{-}energy}(i_1, i_2, i_3, \ldots, x) = \int_0^{i_1} \lambda_1(i_1', 0, 0, \ldots, x)\,\mathrm{d}i_1'$$

$$+ \int_0^{i_2} \lambda_2(i_1, i_2', 0, \ldots, x)\,\mathrm{d}i_2'$$

$$+ \int_0^{i_3} \lambda_3(i_1, i_2, i_3', \ldots, x)\,\mathrm{d}i_3'$$

$$+ \cdots$$

Units: Joules for both E and co-E

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Today

- Nonlinear inductors
- Two-coil inductors
- Bonus: Inductance matrix

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Aside: Inductance matrix

Optional, but makes life significantly easier

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

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Speed voltage (Inductance matrix)

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

Matrix-valued function of θ

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\right) \qquad \boxed{M'(\theta) = \frac{\partial}{\partial\theta}M(\theta)}$$

$$= \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\right)\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix}}_{\text{Transformer voltage}} + \underbrace{\begin{bmatrix}0 & M'(\theta)\\ M'(\theta) & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t}}_{\text{Speed voltage}}$$

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Speed voltage (Inductance matrix)

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \underbrace{\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix}}_{\text{Transformer voltage}} + \underbrace{\begin{bmatrix}0 & M'(\theta)\\ M'(\theta) & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t}}_{\text{Speed voltage}}$$

Example: if $M(\theta) = M_0\cos\theta$, then

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \begin{bmatrix}L_s & M_0\cos\theta\\ M_0\cos\theta & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix} + \begin{bmatrix}0 & -M_0\sin\theta\\ -M_0\sin\theta & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

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Energy and Co-Energy (Inductance matrix)

Theorem. If λ = L(θ)i, then stored energy equals

$$E = \underbrace{\frac{1}{2}i^T L(\theta) i}_{\text{Co-energy}} = \underbrace{\frac{1}{2}\lambda^T L(\theta)^{-1}\lambda}_{\text{Energy}} \quad \text{[J]}$$

Co-energy function of current

Energy function of flux

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Torque, Co-Energy (Inductance matrix)

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

Textbook pp. 110-114

Matrix-valued function of θ

$$\tau^e = +\frac{\partial}{\partial\theta}[\text{Co-energy}] \qquad E = \frac{1}{2}i^T L(\theta) i$$

Co-energy is function of current

$$= \frac{1}{2}i^T L'(\theta) i$$

$$= i_s i_r M'(\theta) \qquad L'(\theta) = \begin{bmatrix}0 & M'(\theta)\\ M'(\theta) & 0\end{bmatrix}$$

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