ECE330: Power Circuits & Electromechanics

Lecture 13. Energy via line integrals

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Schedule

- ~~Wed 3/4: Voltage of mechanical origin~~
- ~~Fri 3/6: Practical transformers (Banerjee)~~
- ~~Mon 3/9: Force of electrical origin~~
- ~~**Wed 3/11: Quiz 5 + Review**~~
- Fri 3/13: Energy via line integrals
- **Mon 3/16: Spring break**
- **Wed 3/18: Spring break**
- **Fri 3/20: Spring break**
- Mon 3/23: Co-energy via line integrals
- **Wed 3/11: Homework 6 + Review**

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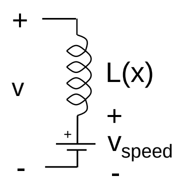
Linear x-varying inductors

Inductance

$$L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Voltage of mechanical origin

$$v = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + [L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}$$



Force of electric origin

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}] = -\frac{\partial}{\partial x}[\text{Energy}]$$

Energy & Co-energy (This week)

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

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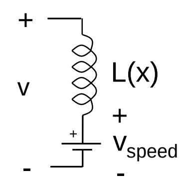
Conservative electromagnetic systems

Flux & current relations

$$\lambda = \lambda(x, i), \quad i = i(\lambda, x)$$

Voltage of mechanical origin

$$v = \underbrace{\frac{\partial \lambda}{\partial i}}_{L(x)}\frac{\mathrm{d}i}{\mathrm{d}t} + \underbrace{\frac{\partial \lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}}_{v_{\text{speed}}}$$



Force of electric origin

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}] = -\frac{\partial}{\partial x}[\text{Energy}]$$

Energy & Co-energy (Today)

$$\text{Co-energy} = \int_0^i \lambda(i', x)\,\mathrm{d}i'$$

$$\text{Energy} = \int_0^\lambda i(\lambda', x)\,\mathrm{d}\lambda'$$

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Today

- (Review) Line integrals
- Line integral definition of energy
- Example: Forces from energy
- Example: Energy from current
- Example: Energy from multi-port current

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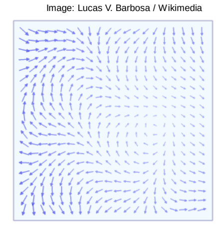
Line integral of a vector field

Image: Lucas V. Barbosa / Wikimedia



Force (vector field)
F: wind force

Position (vector)
a: CMI
b: ORD
C: flight path
r(t): GPS position

Velocity (vector)
r'(t): GPS velocity

Work or energy (scalar)
Line integral: Total work done over path

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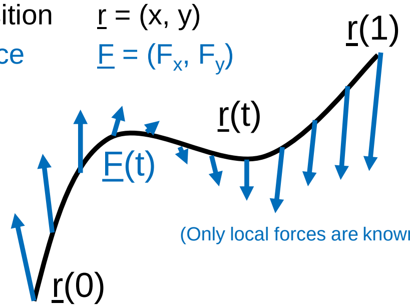
Line integral of a time-varying vector

Position $\underline{r} = (x, y)$

Force $\underline{F} = (F_x, F_y)$



How to compute total work done?

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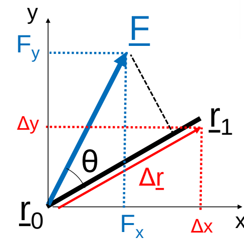
Closed-form solution for a line segment

Position $\underline{r}_0 = (x_0, y_0)$, $\underline{r}_1 = (x_1, y_1)$

Force $\underline{F} = (F_x, F_y)$



$$\Delta\underline{r} = \underline{r}_1 - \underline{r}_0 = (x_1 - x_0, y_1 - y_0) = (\Delta x, \Delta y)$$

$$E = |\underline{F}||\Delta\underline{r}|\cos\theta = \underline{F}\cdot\Delta\underline{r} = (F_x, F_y)\cdot(\Delta x, \Delta y) = F_x\Delta x + F_y\Delta y$$

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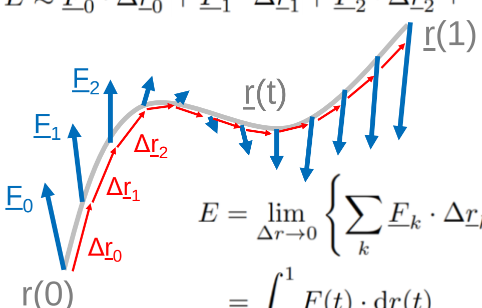
Riemann line integral

$$E \approx \underline{F}_0\cdot\Delta\underline{r}_0 + \underline{F}_1\cdot\Delta\underline{r}_1 + \underline{F}_2\cdot\Delta\underline{r}_2 + \cdots$$



$$E = \lim_{\Delta r\to 0}\left\{\sum_k \underline{F}_k\cdot\Delta\underline{r}_k\right\} = \int_0^1 \underline{F}(t)\cdot \mathrm{d}\underline{r}(t)$$

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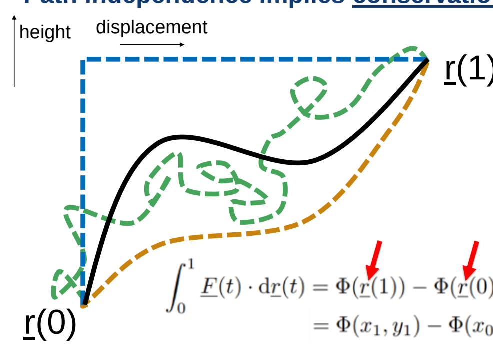
Path independence implies conservation



$$\int_0^1 \underline{F}(t)\cdot\mathrm{d}\underline{r}(t) = \Phi(\underline{r}(1)) - \Phi(\underline{r}(0)) = \Phi(x_1, y_1) - \Phi(x_0, y_0)$$

Conservative energy is a function of position

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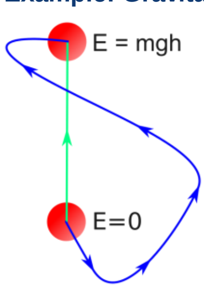
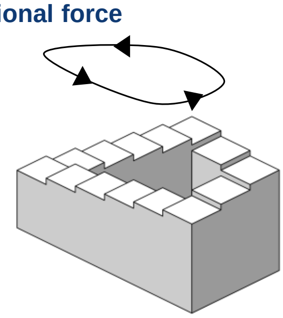
Example: Gravitational force





Path independence: if and only if conservative

Not conservative (Perpetual motion possible)

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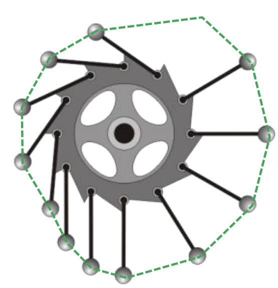
Example: Perpetual motion machines



Zero net energy in a loop

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Today

- ~~(Review) Line integrals~~
- Line integral definition of energy
- Example: Forces from energy
- Example: Energy from current
- Example: Energy from multi-port current

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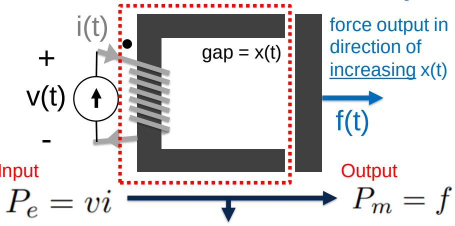
Conservative electromechanical system



$$P_e = vi \qquad P_m = f\frac{dx}{dt}$$

Storage $$\frac{dE}{dt} = i\frac{d\lambda}{dt} - f\frac{dx}{dt}$$

Energy over Δt

$$\Delta E = i\Delta\lambda - f\Delta x$$

Closed system = Conservative

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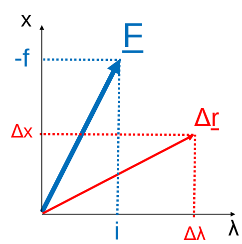
Line integral view of stored energy

$$\begin{aligned}\Delta E &= i\Delta\lambda - f\Delta x \\ &= (i, -f)\cdot(\Delta\lambda, \Delta x) \\ &= \underline{F}\cdot\Delta\underline{r}\end{aligned}$$



Position r = (λ, x)

Force F = (i, -f)

Flux as spatial dim. Current as force.

etymology for "mmf"

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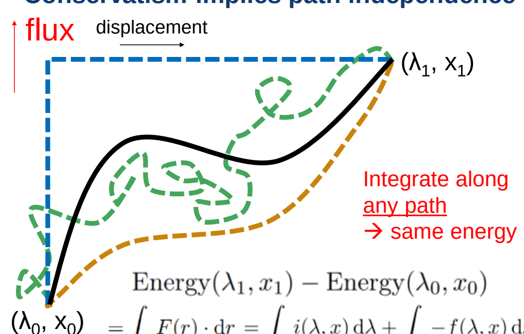
Conservatism implies path independence



$$\mathrm{Energy}(\lambda_1, x_1) - \mathrm{Energy}(\lambda_0, x_0) = \int_C \underline{F}(\underline{r})\cdot d\underline{r} = \int_C i(\lambda, x)\,d\lambda + \int_C -f(\lambda, x)\,dx$$

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Line integral definition of energy

$$\mathrm{Energy}(\lambda_1, x_1) - \mathrm{Energy}(\lambda_0, x_0) = \int_C \underline{F}(\underline{r})\cdot d\underline{r} = \underbrace{\int_C i(\lambda, x)\,d\lambda}_{\text{Energy From Electrical}} + \underbrace{\int_C -f(\lambda, x)\,dx}_{\text{Energy From Mechanical}}$$

Path independent (via conservation)

Path dependent

$$\mathrm{Energy}(0, x) = 0$$ (Can be any constant)

Implies no flux, no force

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Force of electric origin in a conservative magnetic system

Theorem. If energy conserving, then force in the direction of increasing x is

$$f^e = -\frac{\partial}{\partial x}[\mathrm{Energy}]$$

and the current into the dot is

$$i = \frac{\partial}{\partial \lambda}[\mathrm{Energy}]$$

By viewing current as force

Proof. By the gradient theorem

$$\underline{F}(\underline{r}) = (i, -f) = \nabla\mathrm{Energy}(\underline{r})$$
$$\underline{r} = (\lambda, x)$$
$\square$

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Today

- ~~(Review) Line integrals~~
- ~~Line integral definition of energy~~
- Example: Energy from current
- Example: Energy from multi-port current

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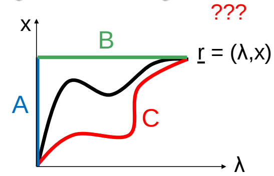
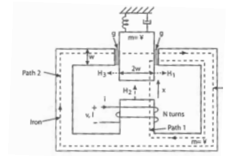
Computing energy from current

$$\mathrm{Energy}(\lambda,x) - \cancel{\mathrm{Energy}(0,0)} \xrightarrow{} 0 \text{ by def}$$

$$= \int_C i(\lambda',x')\,\mathrm{d}\lambda' + \underbrace{\int_C -f(\lambda',x')\,\mathrm{d}x'}_{???}$$



Special integration path avoids the need to know the force



Draw magnetic circuit, compute reluctance & flux, compute flux linkage solve for current

$$i(\lambda,x) = \left(\frac{g+x}{L_0}\right)\lambda$$

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Computing energy from current

$$\mathrm{Energy}(\lambda,x) - \cancel{\mathrm{Energy}(0,0)} \xrightarrow{} 0 \text{ by def}$$

$$= \int_C i(\lambda',x')\,\mathrm{d}\lambda' + \int_C -f(\lambda',x')\,\mathrm{d}x'$$

no change in flux → 0; no flux no force → 0

$$= \left(\cancel{\int_A i(\lambda',x')\,\mathrm{d}\lambda'} + \cancel{\int_A -f(\lambda',x')\,\mathrm{d}x'}\right)$$

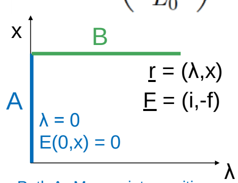
no change in x → 0

$$+ \left(\int_B i(\lambda',x')\,\mathrm{d}\lambda' + \cancel{\int_B -f(\lambda',x')\,\mathrm{d}x'}\right)$$

$$= \int_0^\lambda \left(\frac{g+x}{L_0}\right)\lambda'\,\mathrm{d}\lambda' = \left[\left(\frac{g+x}{L_0}\right)\frac{(\lambda')^2}{2}\right]_0^\lambda$$

$$\boxed{= \left(\frac{g+x}{L_0}\right)\frac{\lambda^2}{2}}$$

$$i(\lambda,x) = \left(\frac{g+x}{L_0}\right)\lambda$$



Path A: Move x into position with no flux
Path B: Charge the inductor with fixed x

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Computing energy from current

$$\mathrm{Energy}(\lambda,x) - \cancel{\mathrm{Energy}(0,0)} \xrightarrow{} 0 \text{ by def}$$

$$= \int_C i(\lambda',x')\,\mathrm{d}\lambda' + \int_C -f(\lambda',x')\,\mathrm{d}x'$$

no change in flux → 0; no flux no force → 0

$$= \left(\cancel{\int_A i(\lambda',x')\,\mathrm{d}\lambda'} + \cancel{\int_A -f(\lambda',x')\,\mathrm{d}x'}\right)$$

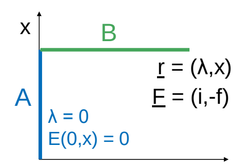
no change in x → 0

$$+ \left(\int_B i(\lambda',x')\,\mathrm{d}\lambda' + \cancel{\int_B -f(\lambda',x')\,\mathrm{d}x'}\right)$$

$$= \int_0^\lambda L(x)^{-1}\lambda'\,\mathrm{d}\lambda' = L(x)^{-1}\left[\frac{(\lambda')^2}{2}\right]_0^\lambda$$

$$\boxed{= \frac{1}{2}L(x)^{-1}\lambda^2}$$

$$i(\lambda,x) = L(x)^{-1}\lambda$$



Path A: Move x into position with no flux
Path B: Charge the inductor with fixed x

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Interpretation as energy from electrical

$$\mathrm{Energy}(\lambda,x) - \cancel{\mathrm{Energy}(0,0)} \xrightarrow{\text{by def}} 0$$

$$= \int_C \underline{F}(\underline{r})\cdot\mathrm{d}\underline{r} = \underbrace{\int_C i(\lambda,x)\,\mathrm{d}\lambda}_{\text{Energy From Electrical}} + \underbrace{\cancel{\int_C -f(\lambda,x)\,\mathrm{d}x}}_{\text{Energy From Mechanical}}$$

by clever choice of path → 0

Path independent (via conservation)

Path dependent

$$\mathrm{Energy}(\lambda,x) = \int_0^\lambda i(\lambda',x)\,\mathrm{d}\lambda'$$

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Energy in a conservative magnetic system

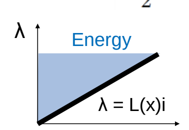
Theorem. If L = ∂λ/∂i > 0, then

$$\mathrm{Energy}(\lambda,x) = \int_0^\lambda i(\lambda',x)\,\mathrm{d}\lambda' \quad [\mathrm{J}]$$

Corollary. If λ = L(x)i and L(x)>0, then

$$\mathrm{Energy}(\lambda,x) = \frac{1}{2}L(x)^{-1}\lambda^2 \quad [\mathrm{J}]$$

Proof.



By definition, energy is the area of the triangle ☐

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