ECE330: Power Circuits & Electromechanics

Lecture 12. Forces of electrical origin

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Schedule

- ~~Wed 3/4: Voltage of mechanical origin~~
- ~~Fri 3/6: Practical transformers (Banerjee)~~
- Mon 3/9: Force of electrical origin
- **Wed 3/11: Quiz 5 + Review**
- Fri 3/13: Energy via line integrals
- **Mon 3/16: Spring break**
- **Wed 3/18: Spring break**
- **Fri 3/20: Spring break**
- Mon 3/23: Banerjee
- **Wed 3/11: Quiz 6 + Review**

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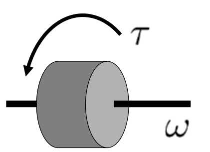
From last lecture (Energy conversion)

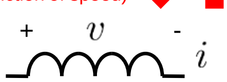


$$\tau = J\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Stores kinetic energy

Voltage of mech. origin (function of speed)

Force of elec. origin (function of current)



$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Stores magnetic energy

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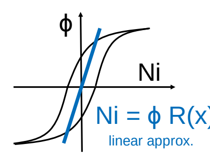
Electrically linear with x-dependence



If electrically linear, then

$$\lambda = L(x)i, \qquad L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Warning: L itself is nonlinear wrt x

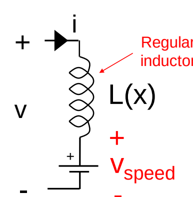
Denote the partial derivative

$$L'(x) \equiv \frac{\partial}{\partial x}L(x)$$

Then,

$$v = \frac{\mathrm{d}}{\mathrm{d}t}[L(x)i]$$

$$= \underbrace{L(x)\frac{\mathrm{d}i}{\mathrm{d}t}}_{\text{Transformer voltage}} + \underbrace{[L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{Speed voltage}}$$



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Today

- Force from conservation of energy
- Translational example
- Rotational example
- Two-coil example

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Important

Stored energy in a linear inductor

Theorem. If λ = L(x)i, then stored energy equals

$$E = \underbrace{\frac{1}{2}L(x)i^2}_{\substack{\textbf{Co-energy}\\ \text{function of current}}} = \underbrace{\frac{1}{2}L(x)^{-1}\lambda^2}_{\substack{\textbf{Energy}\\ \text{function of flux}}} \quad \text{[J]}$$

Remark. Energy = co-energy if and only if **electrically linear**

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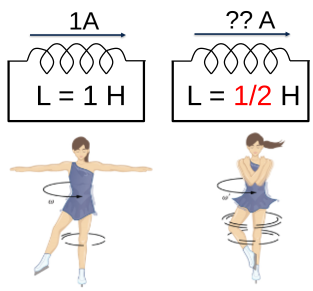
Important

Big picture: Conservation laws



Energy after?
A) 1J; B) 1/2J;
C) 1/4J; D) 1/8J

Voltage = Force
Current = Velocity
Flux = Momentum

No external force → Conservation of momentum
It takes work to change inertia!

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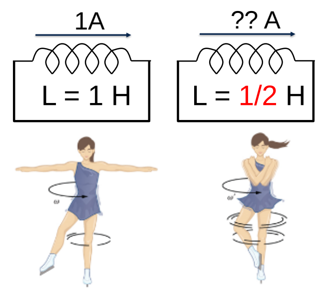
Important

Big picture: Force via conservation laws



Energy after?
A) 1J; B) 1/2J;
C) 1/4J; D) 1/8J

Gap before = 1m
Gap after = 1.1m
Av. Force = ? N

Force direction?
A) Close gap; B) Open gap

No external force → Conservation of momentum
ΔEnergy = work done = force x dist

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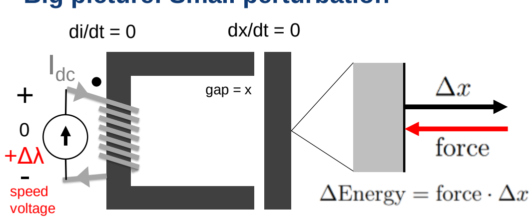
Big picture: Small perturbation



1. Fix system gap x, and fix current I_{dc}.
2. Perturb x ← x + Δx and measure work done ΔE.
3. Force is ΔE / Δx in direction of <u>decreasing</u> x(t).

<u>Not a closed system</u>: change in stored energy ≠ work done

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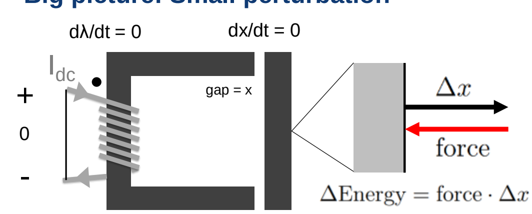
Important

Big picture: Small perturbation



1. Fix system gap x, and fix current I_{dc}.
2. Replace current source by ideal short-circuit.
3. Perturb x ← x + Δx and measure work done ΔE.
4. Force is ΔE / Δx in direction of <u>decreasing</u> x(t).

Closed system: change in stored energy = work done

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Force of electric origin

Theorem. Force in the direction of <u>increasing</u> x is

$$f^e = -\left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{\lambda\ \mathrm{const}} = -\frac{\partial}{\partial x}[\text{Energy}]$$

function of flux

Proof. Conservation of momentum and conservation of energy.

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Force of electric origin

Theorem. Force in the direction of <u>increasing</u> x is

$$f^e = -\left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{\lambda\ \mathrm{const}} = -\frac{\partial}{\partial x}[\text{Energy}]$$

function of flux

$$f^e = +\left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{i\ \mathrm{const}} = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

function of current

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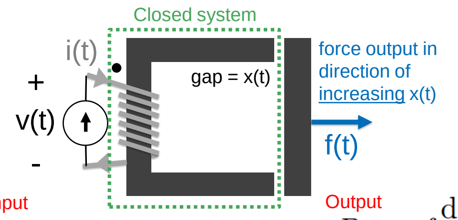
Power flow diagram and polarities



Input

$$P_e = vi$$

Output

$$P_m = f\frac{\mathrm{d}x}{\mathrm{d}t}$$

Storage $\frac{\mathrm{d}E}{\mathrm{d}t} = vi - f\frac{\mathrm{d}x}{\mathrm{d}t}$

Proof idea: Differentiate stored energy

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Force from differentiating energy

$$E = \frac{1}{2}L(x)^{-1}\lambda^2 \qquad \boxed{f^e = -\frac{\partial}{\partial x}[\text{Energy}]}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \frac{\partial E}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= [L(x)^{-1}\lambda]\frac{\mathrm{d}\lambda}{\mathrm{d}t} - \frac{1}{2}L'(x)L(x)^{-2}\lambda^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

= -partial

$$= vi - \frac{1}{2}L'(x)i^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

force of electric origin

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Force from differentiating co-energy

$$E = \frac{1}{2}L(x)i^2 \qquad \boxed{f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial E}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = [L(x)i]\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{2}L'(x)i^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= i\left[L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{2}L'(x)i\frac{\mathrm{d}x}{\mathrm{d}t}\right]$$

= +partial

Transformer voltage

$$= i\left[\left(v - L'(x)i\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{1}{2}L'(x)i\frac{\mathrm{d}x}{\mathrm{d}t}\right]$$

$$= vi - \frac{1}{2}L'(x)i^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

force of electric origin ☐

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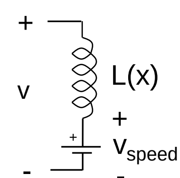
Summary: Linear x-varying inductors

Inductance (Weeks 3-4)

$$L(x) = \frac{N^2}{\mathcal{R}(x)}$$



Voltage of mechanical origin

$$v = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + [L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}$$

Force of electric origin

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

$$= -\frac{\partial}{\partial x}[\text{Energy}]$$

Energy & Co-energy

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

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Today

- ~~Force from conservation of energy~~
- Translational example
- Rotational example
- Two-coil example

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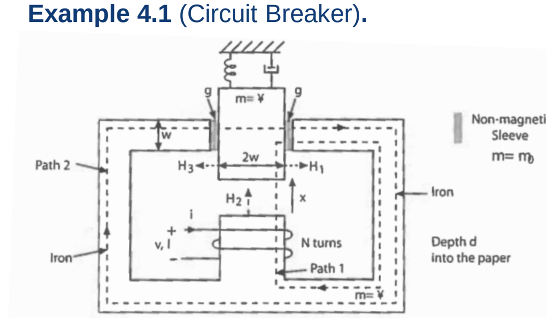
Example 4.1 (Circuit Breaker).



Goal: Find force of electric origin

Approach: Find L(x) and differentiate

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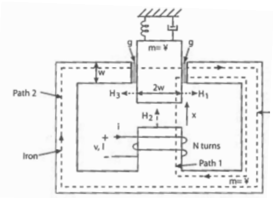
Example 4.1 (Circuit Breaker).



Draw magnetic circuit, compute reluctance & flux, compute flux linkage, extract inductance term

$$L(x) = \frac{2L_0}{g+x}$$

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

$$f^e = ?$$

A) $\cancel{+2L_0/(g+x)}$

B) $-i^2L_0/(g+x)^2$

C) $\cancel{+i^2L_0/(g+x)^2}$

D) $\cancel{gi^2L_0/(g+x)^2}$

E) $\cancel{xi^2L_0/(g+x)^2}$

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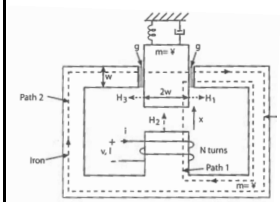
Example 4.1 (Circuit Breaker).



Draw magnetic circuit, compute reluctance & flux, compute flux linkage, extract inductance term

$$L(x) = \frac{2L_0}{g+x}$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

$$f^e = -\frac{\partial}{\partial x}[\text{Energy}]$$

$$f^e = ?$$

A) $-\lambda^2/(4L_0)$

B) $\cancel{-\lambda/(2L_0)}$

C) $\cancel{\lambda(\lambda+2g+2x)/(4L_0)}$

D) $\cancel{-\lambda^2/L_0}$

E) $-\lambda(\lambda+2g+2x)/L_0$

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Example 4.1 (Circuit Breaker).

$$L(x) = \frac{2L_0}{g+x}$$

$$f^e = \frac{-L_0 i^2}{(g+x)^2} = \frac{-1}{4L_0}\frac{4L_0^2 i^2}{(g+x)^2} = \frac{-\lambda^2}{4L_0}$$

From co-energy From energy

Indeed, both co-energy and energy yield the same number

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Today

- ~~Force from conservation of energy~~
- ~~Translational example~~
- Rotational example
- Two-coil example

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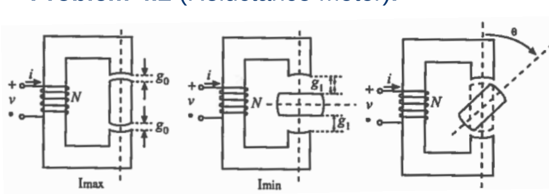
Problem 4.2 (Reluctance motor).



Measure inductance as

$$L(\theta) = L_1 + L_2\cos(2\theta)$$

Goal: Torque of electric origin

Approach: Find L(θ) and differentiate

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Today

- ~~Force from conservation of energy~~
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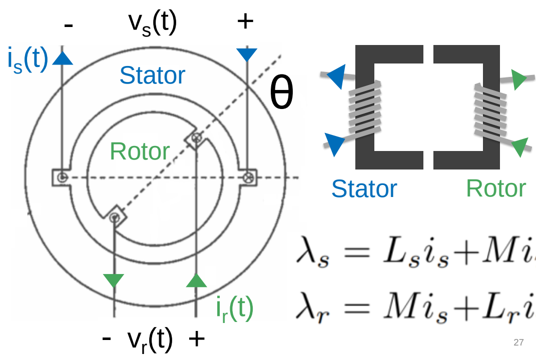
Example 4.2 (Doubly-Fed Machine)



$$\lambda_s = L_s i_s + M i_r$$

$$\lambda_r = M i_s + L_r i_r$$

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Stored energy in a linear inductor (Inductor matrix)

Theorem. If λ = L(θ)i, then stored energy equals

$$E = \frac{1}{2} i^T L(\theta) i \;=\; \frac{1}{2} \lambda^T L(\theta)^{-1} \lambda \quad \text{[J]}$$

Co-energy function of current

Energy function of flux

Proof: Next Monday (?)

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Aside: Matrix quadratic forms

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$$

$$= \begin{matrix} x^T A x + x^T B y \\ + y^T C x + y^T D y \end{matrix}$$

$$x^T M x = \sum_{i=1}^{n} \sum_{j=1}^{n} M[i,j] x[i] x[j]$$

Start brushing up on your linear algebra!

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Example 4.2 (Doubly-Fed Machine)

$$\begin{bmatrix} \lambda_s \\ \lambda_r \end{bmatrix} = \begin{bmatrix} L_s & M(\theta) \\ M(\theta) & L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix}$$

Textbook pp. 110-114

Matrix-valued function of θ

$$\tau^e = +\frac{\partial}{\partial \theta} [\text{Co-energy}] \qquad E = \frac{1}{2} i^T L(\theta) i$$

Co-energy is function of current

$$= \frac{1}{2} i^T L'(\theta) i$$

$$= \frac{1}{2} i_s i_r M'(\theta) \qquad L'(\theta) = \begin{bmatrix} 0 & M'(\theta) \\ M'(\theta) & 0 \end{bmatrix}$$

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Schedule

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