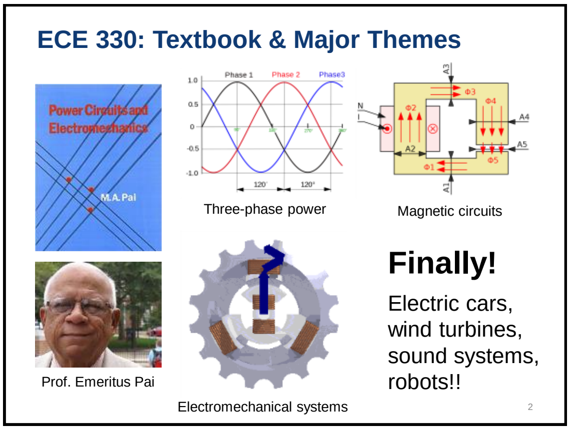
### ECE330: Power Circuits & Electromechanics
## Lecture 11. Voltage of mechanical origin

**Prof. Richard Y. Zhang**
Univ. of Illinois at Urbana-Champaign
ryz@illinois.edu



## ECE 330: Textbook & Major Themes

Three-phase power

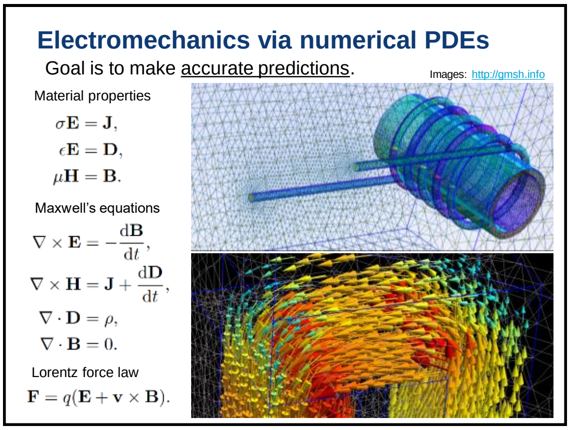
Magnetic circuits

Prof. Emeritus Pai

Electromechanical systems

**Finally!**

Electric cars, wind turbines, sound systems, robots!!



## Electromechanics via numerical PDEs

Goal is to make accurate predictions.

Images: http://gmsh.info

Material properties

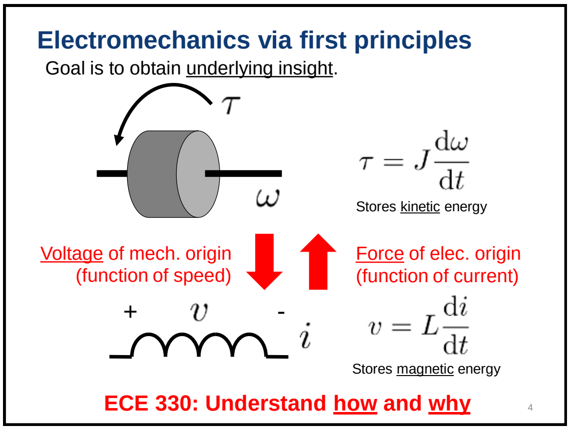
$$\sigma \mathbf{E} = \mathbf{J},$$
$$\epsilon \mathbf{E} = \mathbf{D},$$
$$\mu \mathbf{H} = \mathbf{B}.$$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt},$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt},$$
$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0.$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$



## Electromechanics via first principles

Goal is to obtain underlying insight.

$$\tau = J\frac{d\omega}{dt}$$

Stores kinetic energy

Voltage of mech. origin (function of speed)

Force of elec. origin (function of current)

$$v = L\frac{di}{dt}$$

Stores magnetic energy

**ECE 330: Understand how and why**

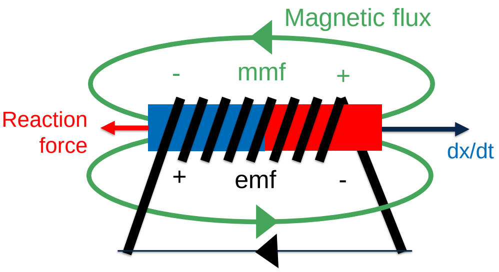
## Schedule

- Wed 3/4: Voltage of mechanical origin
- Fri 3/6: Practical transformers (Banerjee)
- Mon 3/9: Force of electrical origin
- **Wed 3/11: Quiz 5**
- Fri 3/13: Energy and co-energy

## Today

- Transformer voltage and speed voltage
- Translational example
- Rotational example
- Two-coil example

## Faraday's experiment



Induced emf from <u>movement</u> (not current)

Induced emf acts like <u>voltage source</u>

---

## <u>Speed voltage</u> in a reluctance machine

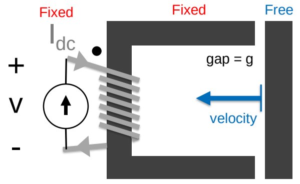


Image: Eurico Zimbres (Wikimedia)

$$\lambda = \frac{N^2 I_{\mathrm{dc}} \leftarrow \text{fixed}}{\mathcal{R}_{\mathrm{core}} + \mathcal{R}_{\mathrm{gap}} \leftarrow \text{free}}$$

$$\mathcal{R}_{\mathrm{gap}} = \frac{2g \leftarrow \text{free}}{\mu_0 w^2}$$

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\mathrm{d}\lambda}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t} = \underbrace{\left[\frac{-N^2 I_{\mathrm{dc}}}{(\mathcal{R}_{\mathrm{core}} + \mathcal{R}_{\mathrm{gap}})^2}\right]}_{\mathrm{d}\lambda/\mathrm{d}\mathcal{R}_{\mathrm{gap}}} \underbrace{\left[\frac{2}{\mu_0 w^2}\right]}_{\mathrm{d}\mathcal{R}_{\mathrm{gap}}/\mathrm{d}g} \frac{\mathrm{d}g}{\mathrm{d}t}$$

sign of v?

v > 0 if dg/dt < 0

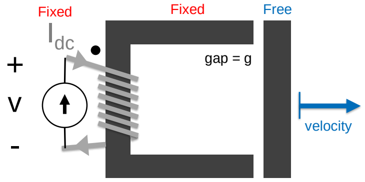
Electric power input.

Mechanical power output?

Not necessarily. Depends on whether energy is <u>stored</u>.

---

## <u>Speed voltage</u> in a reluctance machine



$$\lambda = \frac{N^2 I_{\mathrm{dc}} \leftarrow \text{fixed}}{\mathcal{R}_{\mathrm{core}} + \mathcal{R}_{\mathrm{gap}} \leftarrow \text{free}}$$

$$\mathcal{R}_{\mathrm{gap}} = \frac{2g \leftarrow \text{free}}{\mu_0 w^2}$$

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\mathrm{d}\lambda}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t} = \underbrace{\left[\frac{-N^2 I_{\mathrm{dc}}}{(\mathcal{R}_{\mathrm{core}} + \mathcal{R}_{\mathrm{gap}})^2}\right]}_{\mathrm{d}\lambda/\mathrm{d}\mathcal{R}_{\mathrm{gap}}} \underbrace{\left[\frac{2}{\mu_0 w^2}\right]}_{\mathrm{d}\mathcal{R}_{\mathrm{gap}}/\mathrm{d}g} \frac{\mathrm{d}g}{\mathrm{d}t}$$

sign of v?

v < 0 if dg/dt > 0

Electric power output.

Mechanical power input?

Again, not necessarily. Must compute forces. (Wed)

---

## Review: Different notions of derivatives

<u>Flux linkage</u> as a function of <u>current</u> and <u>position</u>

$$\lambda(x,i) = xi \qquad 0 = x + i$$

← i depends on x

Ordinary derivative

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} =?$$

~~A) $i$~~

B) $2i$

~~C) $xi$~~

~~D) $x^2$~~

Partial derivative

$$\frac{\partial\lambda}{\partial x} =?$$

A) $i$

~~B) $2i$~~

~~C) $xi$~~

~~D) $x^2$~~

$$\frac{\mathrm{d}(xi)}{\mathrm{d}x} = \frac{\mathrm{d}(-x^2)}{\mathrm{d}x} = -2x = 2i$$

Partial derivative treats <u>dependent</u> variables as <u>independent</u>

**Multivariate chain rule**

Memorize, keep track of dep and indep

$$\frac{\mathrm{d}f(t,x_1,\ldots,x_n)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}$$

(Intuitive explanation)

When do the two derivatives coincide?

---

## Transformer voltage & speed voltage

<u>Flux linkage</u> as a function of <u>current</u> and <u>position</u>

$$\lambda(i,x)$$

Time-variance is due to both current and position

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \underbrace{\frac{\partial\lambda}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t}}_{\text{Transformer voltage}} + \underbrace{\frac{\partial\lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{Speed voltage}}$$

← multivariate chain rule

If dx/dt = 0, then recover inductor equation

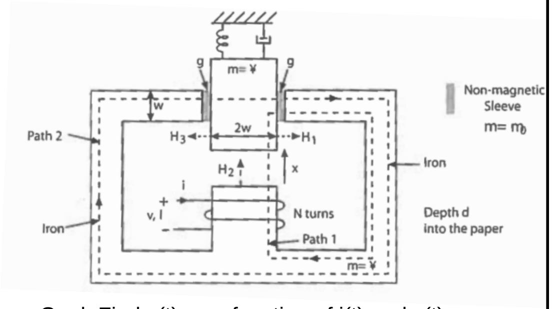
$$v = L(i,x)\frac{\mathrm{d}i}{\mathrm{d}t} \qquad \text{where } L = \frac{\partial\lambda}{\partial i}$$

---

## Today

- ~~Transformer voltage and speed voltage~~
- Translational example
- Rotational example
- Two-coil example

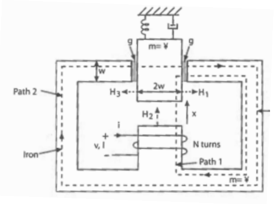
## **Example 4.1** (Circuit Breaker).

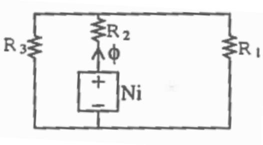


Goal: Find v(t) as a function of i(t) and x(t)

Approach: Find λ(i,x) and differentiate

## **Example 4.1** (Circuit Breaker).





$$\mathcal{R}_1 = \mathcal{R}_3 = \frac{g}{\mu_o wd}$$

$$\mathcal{R}_2 = \frac{x}{\mu_o 2wd}$$

$$\lambda = N\phi = ?$$

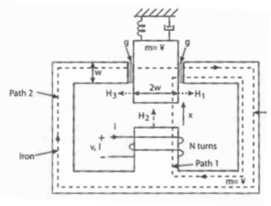
A) $\cancel{\dfrac{Ni}{(\mathcal{R}_1 \parallel \mathcal{R}_3) + \mathcal{R}_2}}$

B) $\cancel{\dfrac{Ni}{\mathcal{R}_1 \parallel \mathcal{R}_2 \parallel \mathcal{R}_3}}$

C) $\cancel{\dfrac{Ni}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3}}$

D) $\dfrac{N^2 i}{(\mathcal{R}_1 \parallel \mathcal{R}_3) + \mathcal{R}_2}$

## **Example 4.1** (Circuit Breaker).



$$\mathcal{R}_1 = \mathcal{R}_3 = \frac{g}{\mu_o wd}$$

$$\mathcal{R}_2 = \frac{x}{\mu_o 2wd}$$

$$\lambda = N\phi = \frac{N^2 i}{(\mathcal{R}_1 \parallel \mathcal{R}_3) + \mathcal{R}_2} = \frac{2wd\mu_0 N^2 i}{g+x}$$

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial\lambda}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial\lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= \underbrace{\left[\frac{2wd\mu_0 N^2}{g+x}\right]\frac{\mathrm{d}i}{\mathrm{d}t}}_{\text{Transformer voltage}} + \underbrace{\left[-\frac{2wd\mu_0 N^2 i}{(g+x)^2}\right]\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{Speed voltage}}$$

## **Example 4.1** (Circuit Breaker).

Flux as a <u>linear function</u> of current

$$\lambda = L(x)i \qquad L(x) = \frac{2wd\mu_0 N^2}{g+x}$$

Then,

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial\lambda}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial\lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= \underbrace{\left[\frac{2wd\mu_0 N^2}{g+x}\right]\frac{\mathrm{d}i}{\mathrm{d}t}}_{\text{Transformer voltage}} + \underbrace{\left[-\frac{2wd\mu_0 N^2 i}{(g+x)^2}\right]\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{Speed voltage}}$$

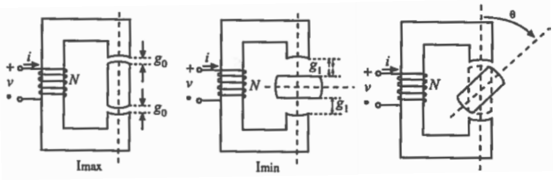
$$= L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + i\frac{\partial L}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

This is an exceptionally common structure!

## Today

- ~~Transformer voltage and speed voltage~~
- ~~Translational example~~
- Rotational example
- Two-coil example

## **Problem 4.2** (Reluctance motor).



Measure inductance as

Why? See homework.

$$L(\theta) = L_1 + L_2\cos(2\theta)$$

Goal: Transformer voltage and speed voltage

Approach: Find λ(i,θ) and differentiate

## **Problem 4.2** (Reluctance motor).

Flux as a linear function of current

$$\lambda = L(\theta)i \qquad L(\theta) = L_1 + L_2\cos(2\theta)$$

Then,

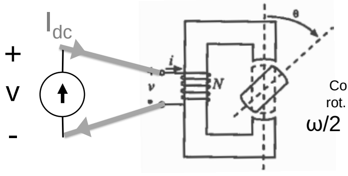
$$v = \frac{d\lambda}{dt} = \frac{\partial\lambda}{\partial i}\frac{di}{dt} + \frac{\partial\lambda}{\partial\theta}\frac{d\theta}{dt}$$

$$= L(\theta)\frac{di}{dt} + i\frac{\partial L}{\partial\theta}\frac{d\theta}{dt}$$

$$v = \underbrace{[L_1 + L_2\cos(2\theta)]\frac{di}{dt}}_{\text{Transformer voltage}} + \underbrace{[-2L_2\sin(2\theta)i]\frac{d\theta}{dt}}_{\text{Speed voltage}}$$

## **Problem 4.2** (Reluctance motor).

$$v = \underbrace{[L_1 + L_2\cos(2\theta)]\frac{di}{dt}}_{\text{Transformer voltage}} + \underbrace{[-2L_2\sin(2\theta)i]\frac{d\theta}{dt}}_{\text{Speed voltage}}$$



Constant rot. speed $\omega/2$ [rad/s]

Let $i(t) = I_{dc}$ and $\theta = (\omega/2)t$

Generates an ac voltage!

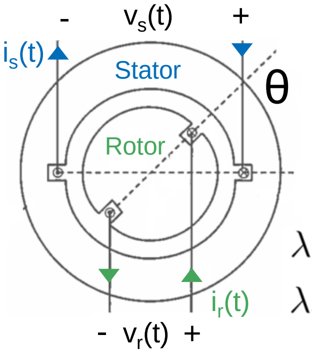
$$v(t) = -\omega L_2 I_{dc}\sin(\omega t)$$

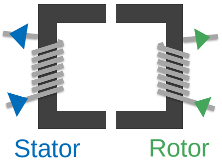
What are some problems?

## **Today**

- ~~Transformer voltage and speed voltage~~
- ~~Translational example~~
- ~~Rotational example~~
- Two-coil example

## **Example 4.2** (Doubly-Fed Machine)

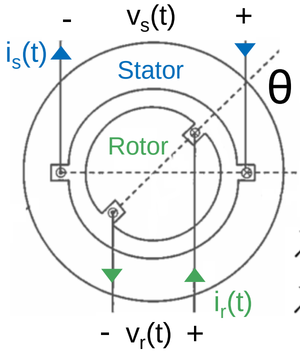




$$\lambda_s = L_s i_s + M i_r$$

$$\lambda_r = M i_s + L_r i_r$$

## **Example 4.2** (Doubly-Fed Machine)



Which term(s) are θ-dependent?

A) Ls, M, Lr Reality

~~B) Ls, M~~

~~C) Lr, M~~

~~D) Ls, Lr~~

E) M Good approximation

Core geometry ~fixed wrt θ.

Coil position free wrt θ.

$$\lambda_s = L_s i_s + M i_r$$

$$\lambda_r = M i_s + L_r i_r$$

## **Example 4.2** (Doubly-Fed Machine)

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

Textbook pp. 110-114

Matrix-valued function of θ

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \frac{d}{dt}\left(\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\right)$$

$$\boxed{M'(\theta) = \frac{\partial}{\partial\theta}M(\theta)}$$

$$= \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{d}{dt}\begin{bmatrix}i_s\\ i_r\end{bmatrix} + \frac{d}{dt}\left(\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\right)\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{d}{dt}\begin{bmatrix}i_s\\ i_r\end{bmatrix}}_{\text{Transformer voltage}} + \underbrace{\begin{bmatrix}0 & M'(\theta)\\ M'(\theta) & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{d\theta}{dt}}_{\text{Speed voltage}}$$

## **Example 4.2** (Doubly-Fed Machine)

$$\begin{bmatrix}\lambda_s\\ \lambda_r\end{bmatrix} = \begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}$$

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \underbrace{\begin{bmatrix}L_s & M(\theta)\\ M(\theta) & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix}}_{\text{Transformer voltage}} + \underbrace{\begin{bmatrix}0 & M'(\theta)\\ M'(\theta) & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t}}_{\text{Speed voltage}}$$

Example: if $M(\theta) = M_0 \cos\theta$, then

Textbook
p. 114
Eqn. 4.35, 4.36

$$\begin{bmatrix}v_s\\ v_r\end{bmatrix} = \begin{bmatrix}L_s & M_0\cos\theta\\ M_0\cos\theta & L_r\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}i_s\\ i_r\end{bmatrix} + \begin{bmatrix}0 & -M_0\sin\theta\\ -M_0\sin\theta & 0\end{bmatrix}\begin{bmatrix}i_s\\ i_r\end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

## Schedule

- ~~Wed 3/4: Voltage of mechanical origin~~
- Fri 3/6: Practical transformers (Banerjee)
- Mon 3/9: Force of electrical origin
- **Wed 3/11: Quiz 5**
- Fri 3/13: Energy and co-energy