sin(x) = cos(x - 90°) \quad \bar{V} = \bar{Z}\bar{I} \quad \bar{S} = \bar{V}\bar{T}^* \quad \mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}

\int_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot n\,da \quad \int_c \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_s \mathbf{B} \cdot n\,da \quad \text{MMF} = Ni = \phi R

\Omega = \frac{1}{\mu A} \quad B = \mu H \quad \phi = BA \quad \lambda = N\phi \quad \lambda = Li (\text{if linear})

\[ \begin{array}{c}
\downarrow \\
N \Rightarrow \\
\uparrow \\
\nu \downarrow \\
\; \\
\end{array} \]

\[ \begin{align*}
W_m &= \int_0^\lambda id\,\lambda \\
W_m' &= \int_0^\lambda d\,i \\
W_m + W_m' &= \lambda i \\
f^e &= -\frac{\partial W_m'}{\partial x} = -\frac{\partial W_m}{\partial x} \quad x \to \Theta \\
\end{align*} \]

\[ \begin{align*}
EFE &= \int_a^b id\,\lambda \\
EFM &= \int_a^b f^e\,dx \\
EFE + EFM &= W_{mb} - W_{ma} \\
\lambda &= \frac{\partial W_m'}{\partial i} \quad i = \frac{\partial W_m}{\partial \lambda}
\end{align*} \]

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \quad \text{and} \quad \dot{x}_2 = f_2(x_1, x_2)
\end{align*} \]

\[ x(t_0 + \Delta t) \approx x(t_0) + \Delta t \cdot \left. \frac{dx}{dt} \right|_{t=t_0} \]
Problem 1 (25 points)

A single-phase generator consists of a coil on the stator and a coil on the rotor with a mutual inductance variation as 0.1\cos(\theta) (Henries) where \theta is the angle from the stator field axis to the rotor field axis. The rotor is being driven at a constant speed of 377 radians per second and the rotor coil has a constant dc current \(i_r = 5\) A. The self inductances of the stator and rotor are both constants and you may assume a linear magnetic core.

a) Compute the open-circuit stator voltage (\(i_s = 0\)) as a function of time (recall that \(d\theta/dt = \omega\) and assume some angle \(\theta = \theta_0\) at time zero)

\[
\lambda_s = L_s i_s + 0.1 \cos(\theta) i_r \\
\lambda_r = 0.1 \cos(\theta) i_s + L_r i_r \\
\theta = 377t + \theta_0 \\

V_s = \frac{d\lambda_s}{dt} = -188.5 \sin(377t + \theta_0) \\
V_r = 188.5 \sin(-[377t + \theta_0]) \\
V_s = 188.5 \cos(377t + \theta_0 - \frac{\pi}{2}) \\
V_s = 188.5 \cos(377t + \theta_0 + \frac{\pi}{2}) V
\]

b) What is the torque of electrical origin when \(i_s = 10\) Amps and \(\theta = 45^\circ\)?

\[
T^e = \frac{\partial W'_m}{\partial \theta} \\
W'_m = \int_0^{i_s} \lambda_s(i_s, i_r = 0, \theta) di_s + \int_0^{i_r} \lambda_r(i_s, i_r, \theta) di_r \\
= \int_0^{i_s} L_s i_s^2 di_s + \int_0^{i_r} (0.1 \cos(\theta) i_s + L_r i_r) di_r \\
= \frac{1}{2} L_s i_s^2 + 0.1 \cos(\theta) i_s i_r + \frac{1}{2} L_r i_r^2 \\
T^e = \frac{\partial W'_m}{\partial \theta} = T^e = -0.1 \sin(\theta) i_s i_r \\
T^e = -0.1(10)(5) \sin(45^\circ) = -3.536 \text{ Nm}
\]

(There is an extra page at the end if you need it)
Problem 2 (25 Points)

The flux linkages for a 2-coil system are given as

\[
\lambda_1 = \left( \frac{0.0085}{1 + \frac{x}{L}} \right) i_1 - \left( \frac{0.0021}{1 + \frac{x}{L}} \right) i_2
\]

\[
\lambda_2 = -\left( \frac{0.0021}{1 + \frac{x}{L}} \right) i_1 + \left( \frac{0.0021}{1 + \frac{x}{L}} \right) i_2
\]

For this system, find:

a) The co-energy. (9 points)

b) The force of electric origin. (9 points)

c) What is the force of electric origin when \(i_1=1\) A, \(i_2=0.5\) A, \(x=1\) cm, and \(L=20\) cm? (7 points)

\[
W_m' = \int \lambda_1(\hat{c}_1, \hat{c}_2) d\hat{c}_1 + \int \lambda_2(\hat{c}_1, \hat{c}_2) d\hat{c}_2
\]

\[
= \int \left( \frac{0.0085}{1 + \frac{x}{L}} \right) c_1^2 d\hat{c}_1 + \int \left( - \frac{0.0021}{1 + \frac{x}{L}} \right) c_2^2 d\hat{c}_2
\]

\[
= \frac{1}{2} \left( \frac{0.0085}{1 + \frac{x}{L}} \right)^2 c_1^2 + \frac{1}{2} \left( \frac{0.0021}{1 + \frac{x}{L}} \right)^2 c_2^2 - \left( \frac{0.0021}{1 + \frac{x}{L}} \right) c_1 c_2
\]

\[
W_m' = \left( \frac{0.00425}{1 + \frac{x}{L}} \right) c_1^2 + \left( \frac{0.00105}{1 + \frac{x}{L}} \right) c_2^2 - \left( \frac{0.0021}{1 + \frac{x}{L}} \right) c_1 c_2
\]

\[
n_e = \frac{\partial W_m'}{\partial x} = \begin{pmatrix} 0.00425 & -0.0021 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\]

\[
f_e = -\frac{1}{L} \left( \frac{0.00425}{1 + \frac{x}{L}} \right) c_1^2 + \frac{1}{L} \left( \frac{0.00105}{1 + \frac{x}{L}} \right) c_2^2 + \frac{1}{L} \left( \frac{0.0021}{1 + \frac{x}{L}} \right) c_1 c_2
\]

\[
f_e = \frac{1}{0.2 \left( 1 + \frac{0.01}{0.2} \right)} \left( 0.00425 c_1^2 + 0.00105 c_2^2 - 0.0021 c_1 c_2 \right)
\]

\[
f_e = -0.0157 \text{ N}
\]

(There is an extra page at the end if you need it)
Problem 3 (25 pts.)

An electromechanical system with $\lambda = L(x)i$ is operated through the transition from a to b as shown below - note that the flux linkage at a is equal to the flux linkage at b:

![Diagram of electromechanical system]

a) Find the energy transferred from the electrical system into the coupling field as the system moves from a to b as shown. (give a graphical answer)
b) Find the energy transferred from the mechanical system into the coupling field as the system moves from a to b as shown (give a graphical answer)
c) If the device moves back from b to a along a constant flux linkage path, find the energy transferred from the electrical system into the coupling field for this motion and the energy transferred from the mechanical system into the coupling field during this motion.
d) If you consider the combined motion of a) to c) above to be one cycle, is this a motor or a generator? Explain why.

\[ a) \quad EFE = \int_{a}^{b} i(\lambda, x) \, d\lambda = \lambda_a - \lambda_b \]

\[ b) \quad \Delta W_m = EFE + EFM \Rightarrow EFM = \Delta W_m - EFE \]

\[ c) \quad EFE = 0 \quad (EFM > 0) \]

(There is an extra page at the end if you need it)
Problem 4 (25 points)

The equation of motion governing the angle of a pendulum connected to a torsional spring under gravity is given as

\[
\frac{d^2 \theta}{dt^2} = -\frac{\kappa}{ml^2} \theta + \frac{g}{l} \sin(\theta)
\]

For this given system:

a) Write the equation of motion in state space form. (7 points)

b) Write the equation(s) you would have to solve (BUT DO NOT SOLVE) for determining the equilibrium positions for the pendulum. (7 points)

c) The function \( \frac{\sin(\theta)}{\theta} \approx 1 - \frac{\theta^2}{6} \). Using this approximation, what are the equilibrium positions for the pendulum? (7 points)

d) Assuming that the pendulum starts at an initial position of \( \theta = 0.01 \) rad, \( \dot{\theta} = 0 \) rad/s, \( m=1 \) kg, \( l=0.1 \) m, \( \kappa = 10 \) Nm/rad, and \( g=9.8 \) m/s\(^2\), find the state variables at \( t=0.002 \) s using a step size \( dt = 0.001 \) s. (4 points)

\[
\begin{align*}
\dot{x}_1 &= \frac{\kappa}{m} x_2 \\
\dot{x}_2 &= \frac{g}{l} \sin(x_1)
\end{align*}
\]

(There is an extra page at the end if you need it)
\[ \frac{dx_1}{dt} = x_2, \]
\[ \frac{x_1^n - x_1^{n-1}}{\Delta t} = x_2^{n-1} \]
\[ x_1^n = x_1^{n-1} + \Delta t x_2^{n-1} \]
\[ \frac{dx_2}{dt} = -\frac{k}{ml^2} x_1 + \frac{g}{l} \sin(x_1) \]
\[ \frac{x_2^n - x_2^{n-1}}{\Delta t} = -\frac{k}{ml^2} x_1^{n-1} + \frac{g}{l} \sin(x_1^{n-1}) \]

\[ \frac{k}{l} = \frac{10}{1(0.05)^2} = 1000 \]
\[ \frac{g}{l} = \frac{9.8}{0.1} = 98 \]

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