ECE330: Power Circuits & Electromechanics

Lecture 12. Forces of electrical origin

Prof. Richard Y. Zhang
Univ. of Illinois at Urbana-Champaign
ryz@illinois.edu

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Schedule

- ~~Mon 10/7: Voltage of mechanical origin~~
- Wed 10/9: Force of electrical origin
- **Friday 10/11: Review + Quiz 5 (Mech. volt.)**
- Mon 10/14: Energy and co-energy
- Wed 10/16: Review
- **Friday 10/18: Review + Quiz 6 (Elec. force)**

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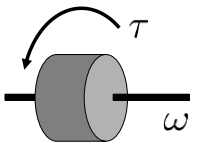
From last lecture (Energy conversion)



$$\tau = J\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Stores kinetic energy

Voltage of mech. origin (function of speed)

Force of elec. origin (function of current)

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Stores magnetic energy

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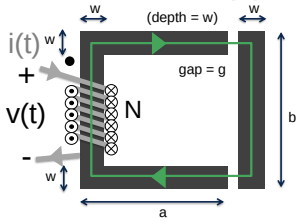
From last lecture (Speed voltage)



$$\lambda = \frac{N^2 i}{\mathcal{R}_{\text{core}} + \mathcal{R}_{\text{gap}}(g)}$$ ← free ← free

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \underbrace{\frac{\partial\lambda}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t}}_{\text{Transformer}} + \underbrace{\frac{\partial\lambda}{\partial g}\frac{\mathrm{d}g}{\mathrm{d}t}}_{\text{Speed}}$$

$$\mathcal{R}_{\text{core}} = \frac{1}{\mu_0\mu_r}\frac{\text{av. path len.}}{\text{cross-sec. area}} = \frac{2a+2b-2w}{\mu_0\mu_r w^2}$$

$$\mathcal{R}_{\text{gap}} = \frac{2g}{\mu_0 w^2}$$ ← free

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Speed voltage in a reluctance machine

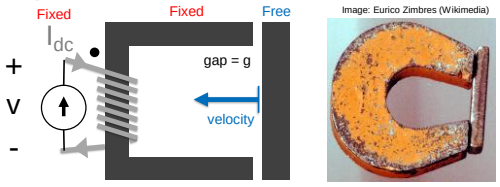

Image: Eurico Zimbres (Wikimedia)

$$v = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\mathrm{d}\lambda}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t} = \underbrace{\left[\frac{-N^2 I_{\text{dc}}}{(\mathcal{R}_{\text{core}}+\mathcal{R}_{\text{gap}})^2}\right]}_{\mathrm{d}\lambda/\mathrm{d}\mathcal{R}_{\text{gap}}}\underbrace{\left[\frac{2}{\mu_0 w^2}\right]}_{\mathrm{d}\mathcal{R}_{\text{gap}}/\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t}$$

v > 0 if dg/dt < 0

Electric power input.

Mechanical power output?

Not necessarily. Depends on whether energy is stored.

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Today

- Force from conservation of energy
- Translational example
- Rotational example
- Two-coil example

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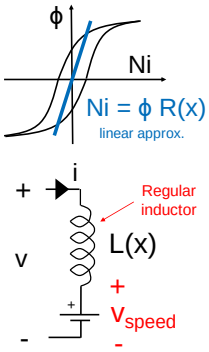
Linear inductor with x-dependence



If linear reluctance, then

$$\lambda = L(x)i, \qquad L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Warning: L itself is nonlinear wrt x

Denote the partial derivative

$$L'(x) \equiv \frac{\partial}{\partial x} L(x)$$

Then,

$$v = \frac{\mathrm{d}}{\mathrm{d}t}[L(x)i]$$

$$= L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + [L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}$$

Transformer voltage | Speed voltage

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Stored energy in a linear inductor

Theorem. If λ = L(x)i, then stored energy equals

$$E = \frac{1}{2}L(x)i^2 \quad = \frac{1}{2}L(x)^{-1}\lambda^2 \quad \text{[J]}$$

Co-energy function of current | **Energy** function of flux

Remark. Energy = co-energy if and only if λ = L(x)i.

Proof: Next Monday

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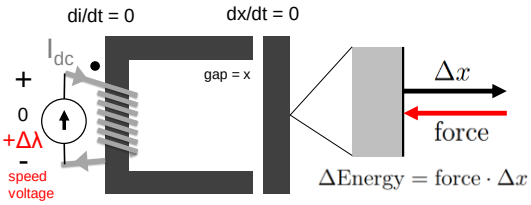
Big picture: Small perturbation



1. Fix system gap x, and fix current I_{dc}.
2. Perturb x ← x + Δx and measure work done ΔE.
3. Force is ΔE / Δx in direction of decreasing x(t).

Not a closed system: change in stored energy ≠ work done

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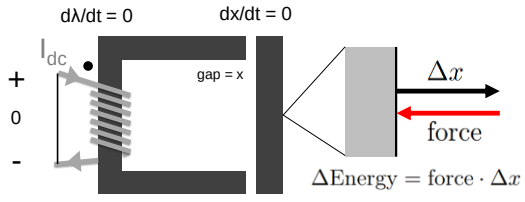
Big picture: Small perturbation



1. Fix system gap x, and fix current I_{dc}.
2. Replace current source by ideal short-circuit.
3. Perturb x ← x + Δx and measure work done ΔE.
4. Force is ΔE / Δx in direction of decreasing x(t).

Closed system: change in stored energy = work done

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Force of electric origin in a linear inductor

Theorem. If λ = L(x)i, then force in the direction of increasing x is

$$f^e = -\left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{\lambda\ \mathrm{const}} = -\frac{\partial}{\partial x}\,[\mathrm{Energy}]$$

function of flux

$$f^e = +\left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{i\ \mathrm{const}} = +\frac{\partial}{\partial x}\,[\mathrm{Co\text{-}energy}]$$

function of current

Remark. Replace x by θ for rotating system.

Remark. Also valid for L(x) matrix.

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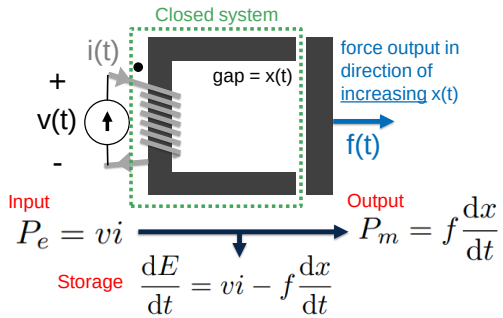
Power flow diagram and polarities



Proof idea: Differentiate stored energy

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Force from differentiating energy

$$E = \frac{1}{2}L(x)^{-1}\lambda^2 \qquad f^e = -\frac{\partial}{\partial x}\,[\text{Energy}]$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial E}{\partial x}\frac{dx}{dt}$$

$$= [L(x)^{-1}\lambda]\frac{d\lambda}{dt} - \frac{1}{2}L'(x)L(x)^{-2}\lambda^2\frac{dx}{dt}$$

= -partial

$$= vi - \frac{1}{2}L'(x)i^2\frac{dx}{dt}$$

force of electric origin

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Force from differentiating co-energy

$$E = \frac{1}{2}L(x)i^2 \qquad f^e = +\frac{\partial}{\partial x}\,[\text{Co-energy}]$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial i}\frac{di}{dt} + \frac{\partial E}{\partial x}\frac{dx}{dt} = [L(x)i]\frac{di}{dt} + \frac{1}{2}L'(x)i^2\frac{dx}{dt}$$

= +partial

$$= i\left[L(x)\frac{di}{dt} + \frac{1}{2}L'(x)i\frac{dx}{dt}\right]$$

Transformer voltage

$$= i\left[\left(v - L'(x)i\frac{dx}{dt}\right) + \frac{1}{2}L'(x)i\frac{dx}{dt}\right]$$

$$= vi - \frac{1}{2}L'(x)i^2\frac{dx}{dt}$$

force of electric origin

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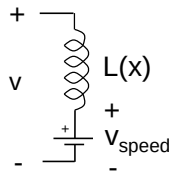
Summary: Linear x-varying inductors

Inductance (Weeks 3-4)

$$L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Voltage of mechanical origin (Mon)

$$v = L(x)\frac{di}{dt} + [L'(x)i]\frac{dx}{dt}$$



Force of electric origin (Today)

$$f^e = +\frac{\partial}{\partial x}\,[\text{Co-energy}]$$

$$= -\frac{\partial}{\partial x}\,[\text{Energy}]$$

Energy & Co-energy (Next Mon)

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

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Today

- ~~Force from conservation of energy~~
- Translational example
- Rotational example
- Two-coil example

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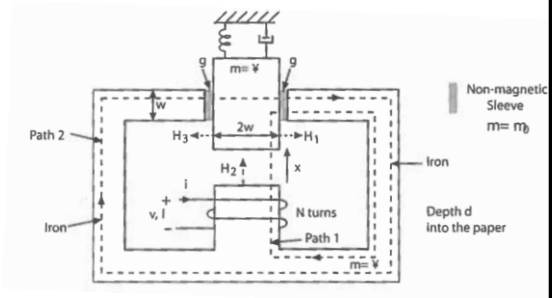
Example 4.1 (Circuit Breaker).



Goal: Find force of electric origin

Approach: Find L(x) and differentiate

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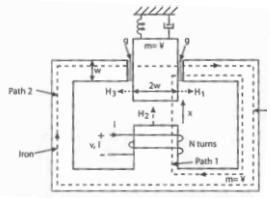
Example 4.1 (Circuit Breaker).



Draw magnetic circuit, compute reluctance & flux, compute flux linkage, extract inductance term

$$L(x) = \frac{2L_0}{g+x}$$

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$f^e = +\frac{\partial}{\partial x}\,[\text{Co-energy}]$$

$f^e = ?$

A) ~~$+2L_0/(g+x)$~~

B) $-i^2L_0/(g+x)^2$

C) ~~$+i^2L_0/(g+x)^2$~~

D) ~~$gi^2L_0/(g+x)^2$~~

E) ~~$xi^2L_0/(g+x)^2$~~

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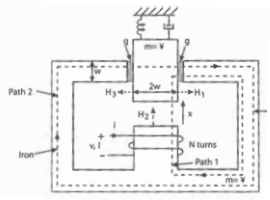
Example 4.1 (Circuit Breaker).



Draw magnetic circuit, compute reluctance & flux, compute flux linkage, extract inductance term

$$L(x) = \frac{2L_0}{g+x}$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

$$f^e = -\frac{\partial}{\partial x}[\text{Energy}]$$

$f^e = ?$

A) $-\lambda^2/(4L_0)$

~~B) $-\lambda/(2L_0)$~~

~~C) $-\lambda(\lambda+2g+2x)/(4L_0)$~~

~~D) $-\lambda^2/L_0$~~

~~E) $-\lambda(\lambda+2g+2x)/L_0$~~

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Example 4.1 (Circuit Breaker).

$$L(x) = \frac{2L_0}{g+x}$$

$$f^e = \frac{-L_0 i^2}{(g+x)^2} = \frac{-1}{4L_0}\frac{4L_0^2 i^2}{(g+x)^2} = \frac{-\lambda^2}{4L_0}$$

From co-energy — From energy

$$f^e = \frac{i^2}{2}L'(x) = \frac{1}{2}L'(x)L(x)^{-2}\lambda^2$$

$$= \frac{\lambda^2}{2}\frac{\partial}{\partial x}[L(x)^{-1}]$$

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Today

- ~~Force from conservation of energy~~
- ~~Translational example~~
- Rotational example
- Two-coil example

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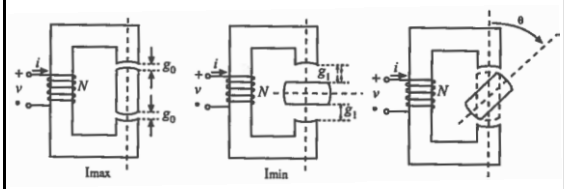
Problem 4.2 (Reluctance motor).



Measure inductance as

$$L(\theta) = L_1 + L_2\cos(2\theta)$$

Goal: Torque of electric origin

Approach: Find L(θ) and differentiate

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Today

- ~~Force from conservation of energy~~
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- ~~Rotational example~~
- Two-coil example

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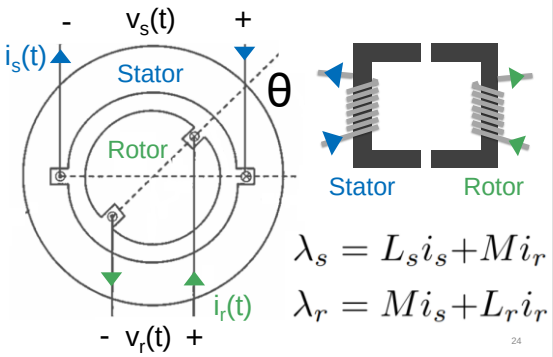
Example 4.2 (Doubly-Fed Machine)



$$\lambda_s = L_s i_s + M i_r$$

$$\lambda_r = M i_s + L_r i_r$$

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Stored energy in a linear inductor (Inductor matrix)

Theorem. If λ = L(θ)i, then stored energy equals

$$E = \frac{1}{2} i^T L(\theta) i \quad = \frac{1}{2} \lambda^T L(\theta)^{-1} \lambda \ \text{[J]}$$

Co-energy function of current — **Energy** function of flux

Proof: Next Monday (?)

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Aside: Matrix quadratic forms

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$$

$$= \begin{array}{l} x^T Ax + x^T By \\ + y^T Cx + y^T Dy \end{array}$$

$$x^T M x = \sum_{i=1}^{n} \sum_{j=1}^{n} M[i,j] x[i] x[j]$$

Start brushing up on your linear algebra!

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Example 4.2 (Doubly-Fed Machine)

$$\begin{bmatrix} \lambda_s \\ \lambda_r \end{bmatrix} = \begin{bmatrix} L_s & M(\theta) \\ M(\theta) & L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix}$$

Textbook pp. 110-114

Matrix-valued function of θ

$$\tau^e = +\frac{\partial}{\partial \theta} [\text{Co-energy}] \qquad E = \frac{1}{2} i^T L(\theta) i$$

Co-energy is function of current

$$= \frac{1}{2} i^T L'(\theta) i$$

$$= \frac{1}{2} i_s i_r M'(\theta) \qquad L'(\theta) = \begin{bmatrix} 0 & M'(\theta) \\ M'(\theta) & 0 \end{bmatrix}$$

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