Review Agenda

- Voltage, forceds, energy, co-energy
- Graphical interpretation
- Multi-port devices
- Spring 2019, Exam 2, Probs 1-2
- Spring 2012, Exam 2, Prob 1

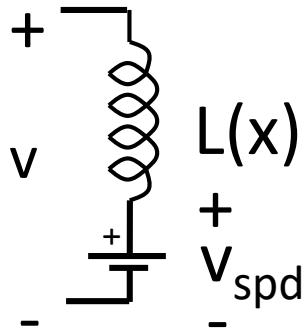
Linear x-varying inductors

Inductance

$$\lambda(i,x) = L(x)i, \qquad L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Voltage of mechanical origin

$$v = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + [L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}$$



Energy & Co-energy

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

function of current

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

function of flux

Force of electric origin

(direction of increasing x)

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

$$= -\frac{\partial}{\partial x}[\text{Energy}]$$

$$\lambda = +\frac{\partial}{\partial i}[\text{Co-energy}]$$

$$i = \frac{\partial}{\partial \lambda}[\text{Energy}]$$

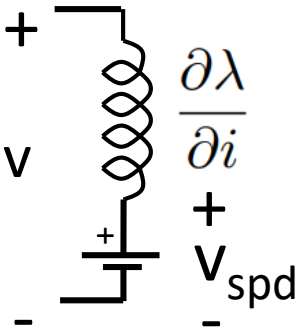
Conservative electromagnetic systems

Flux & current relations

$$\lambda = \lambda(i, x), \;\; i = i(\lambda, x)$$

Voltage of mechanical origin

$$v = \frac{\partial \lambda}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial \lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$



Energy & Co-energy

$$\text{Co-energy} = \int_0^i \lambda(i', x)\,\mathrm{d}i'$$

function of current

$$\text{Energy} = \int_0^\lambda i(\lambda', x)\,\mathrm{d}\lambda'$$

function of flux

$$\text{Co-energy}(i, x) = \lambda i - \text{Energy}(\lambda, x)$$

Force of electric origin

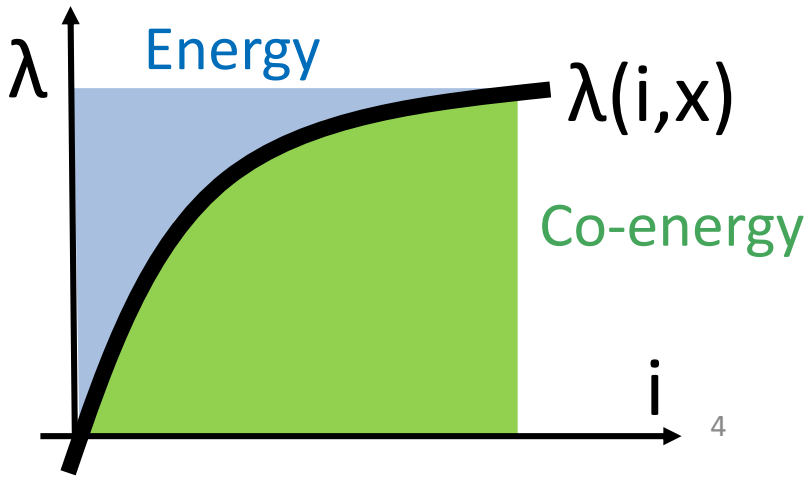
(direction of increasing x)

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

$$= -\frac{\partial}{\partial x}[\text{Energy}]$$

$$\lambda = +\frac{\partial}{\partial i}[\text{Co-energy}]$$

$$i = \frac{\partial}{\partial \lambda}[\text{Energy}]$$

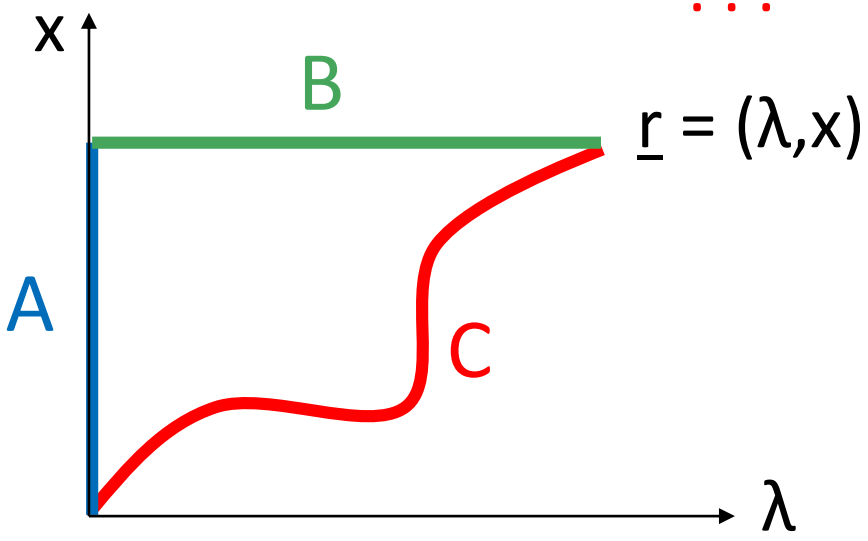


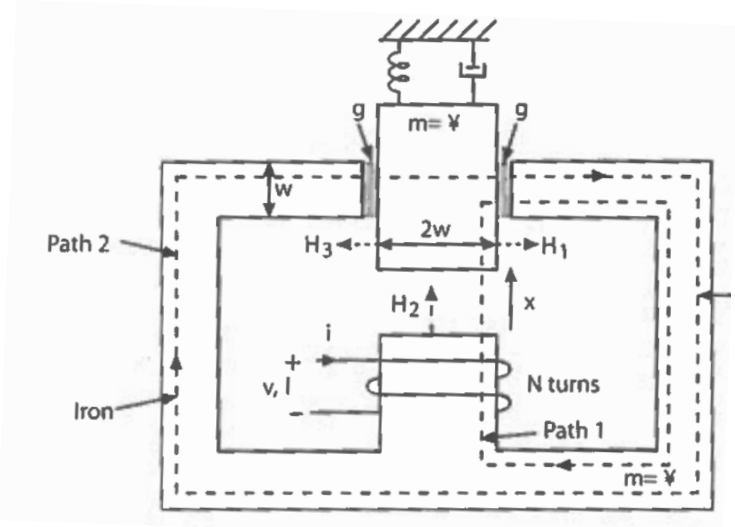
Derivation of the energy integral

$$\text{Energy}(\lambda, x) - \cancel{\text{Energy}(0,0)} = \int_C i(\lambda', x')\,\mathrm{d}\lambda' + \int_C -f(\lambda', x')\,\mathrm{d}x'$$

0 by def

???





Draw magnetic circuit,
compute reluctance & flux,
compute flux linkage
solve for current

Special integration path avoids the need to know the force

$$i(\lambda, x) = \left(\frac{g + x}{L_0}\right)\lambda$$

Energy in a conservative magnetic system with multiple ports

always fixed

Theorem. If $\underline{\underline{L}} = \nabla_{\underline{i}} \underline{\lambda} \succ 0$, then

$$\begin{aligned}\text{Energy}(\lambda_1, \lambda_2, \lambda_3 \ldots, x) = &\int_0^{\lambda_1} i_1(\lambda_1', 0, 0, \ldots, x)\, \mathrm{d}\lambda_1' \\ &+ \int_0^{\lambda_2} i_2(\lambda_1, \lambda_2', 0, \ldots, x)\, \mathrm{d}\lambda_2' \\ &+ \int_0^{\lambda_3} i_3(\lambda_1, \lambda_2, \lambda_3', \ldots, x)\, \mathrm{d}\lambda_3' \\ &+ \cdots \quad \text{[J]}\end{aligned}$$

"Charge one coil at a time"

Corollary. If $\underline{\lambda} = \underline{\underline{L}}(x)\underline{i}$ and $\underline{\underline{L}}(x) \succ 0$, then

$$\text{Energy}(\underline{\lambda}, x) = \frac{1}{2}\underline{\lambda}^T \underline{\underline{L}}(x)^{-1} \underline{\lambda} \quad \text{[J]}$$

Co-energy in a conservative magnetic system with multiple ports

always fixed

Theorem. If $\underline{\underline{L}} = \nabla_{\underline{i}} \underline{\lambda} \succ 0$, then

$$\text{Co-energy}(i_1, i_2, i_3, \ldots, x) = \int_0^{i_1} \lambda_1(i_1', 0, 0, \ldots, x)\, \mathrm{d}i_1'$$

"Charge one coil at a time"

$$+ \int_0^{i_2} \lambda_2(i_1, i_2', 0, \ldots, x)\, \mathrm{d}i_2'$$

$$+ \int_0^{i_3} \lambda_3(i_1, i_2, i_3', \ldots, x)\, \mathrm{d}i_3'$$

$$+ \cdots$$

Corollary. If $\underline{\lambda} = \underline{\underline{L}}(x)\underline{i}$ and $\underline{\underline{L}}(x) \succ 0$, then

$$\text{Co-energy}(\underline{i}, x) = \frac{1}{2}\underline{i}^T \underline{\underline{L}}(x)\underline{i} \qquad \text{[J]}$$

Problem 1 (25 points)

A single-phase generator consists of a coil on the stator and a coil on the rotor with a mutual inductance variation as $0.1\cos(\theta)$ (Henries) where θ is the angle from the stator field axis to the rotor field axis . The rotor is being driven at a constant speed of 377 radians per second and the rotor coil has a constant dc current $i_r = 5$ A. The self inductances of the stator and rotor are both constants and you may assume a linear magnetic core.

a) Compute the open-circuit stator voltage ($i_s = 0$) as a function of time (recall that $d\theta/dt=\omega$ and assume some angle $\theta= \theta_o$ at time zero)

$$\lambda_s = L_s i_s + 0.1\cos(\theta) i_r$$
$$\lambda_r = 0.1\cos(\theta) i_s + L_r i_r$$
$$\theta = 377t + \theta_o$$

$$V_s = \frac{d\lambda_s}{dt}$$
$$V_s = L_s \frac{di_s}{dt}^{\to 0} + 0.1\cos(\theta)\frac{di_r}{dt}^{\to 0} - 0.1\sin(\theta)\frac{d\theta}{dt} i_r$$

b) What is the torque of electrical origin when $i_s = 10$ Amps and $\theta = 45°$?

$$T^e = \frac{\partial W_m'}{\partial \theta}$$

$$W_m' = \int_0^{i_s} \lambda_s(\hat{i}_s, i_r=0, \theta)\, d\hat{i}_s + \int_0^{i_r} \lambda_r(i_s, \hat{i}_r, \theta)\, d\hat{i}_r$$

Problem 2 (25 Points)

The flux linkages for a 2-coil system are given as

$$\lambda_1 = \left(\frac{0.0085}{1+\frac{x}{l}}\right) i_1 - \left(\frac{0.0021}{1+\frac{x}{l}}\right) i_2$$

$$\lambda_2 = -\left(\frac{0.0021}{1+\frac{x}{l}}\right) i_1 + \left(\frac{0.0021}{1+\frac{x}{l}}\right) i_2$$

For this system, find:

a) The co-energy. (9 points)
b) The force of electric origin. (9 points)
c) What is the force of electric origin when i_1=1 A, i_2=0.5 A, x=1 cm, and l=20 cm? (7 points)

$$W_m' = \int_0^{i_1} \lambda_1(\hat{i}_1, i_2=0, x)\, d\hat{i}_1 + \int_0^{i_2} \lambda_2(i_1, \hat{i}_2, x)\, d\hat{i}_2$$

$$f^e = \frac{\partial W_m'}{\partial x}$$

Problem 1 [25 points]

The flux linkages in an energy-conservative electromechanical system are given by

$$\lambda_a = L_a i_a + (M\cos\theta) i_b,$$
$$\lambda_b = L_b i_b + (M\cos\theta) i_a + (M\sin\theta) i_c,$$
$$\lambda_c = L_c i_c + (M\sin\theta) i_b,$$

where L_a, L_b, L_c and M are positive constants, and i_a, i_b, i_c are currents into the system.

(a) Is the system electrically linear? [1 point]

(b) How many electrical and mechanical ports does the system have? [2 + 2 points]

(c) Find the co-energy $W'_m(i_a, i_b, i_c, \theta)$ for this system. [12 points]

(d) Compute the torque of electric origin $T_e(i_a, i_b, i_c, \theta)$. [3 points]

C. $$W_m'(i_a, i_b, i_c, \theta) = \int_0^{i_a} \lambda_a(i_a', 0, 0, \theta)\, di_a' + \int_0^{i_b} \lambda_b(i_a, i_b', 0, \theta)\, di_b' + \int_0^{i_c} \lambda_c(i_a, i_b, i_c', \theta)\, di_c'$$

$$W_m'(i_a, i_b, i_c, \theta) = \frac{1}{2} L_a i_a^2 + \frac{1}{2} L_b i_b^2 + M\cos(\theta)\, i_a i_b + \frac{1}{2} L_c i_c^2 + M\sin(\theta)\, i_b i_c$$

D. $$T_e(i_a, i_b, i_c, \theta) = \frac{\partial W'(i_a, i_b, i_c, \theta)}{\partial \theta}$$

$$= -M\sin(\theta)\, i_a i_b + M\cos(\theta)\, i_b i_c$$