USEFUL INFORMATION

\[ \oint \mathbf{H} \cdot d\mathbf{E} = \int_S \mathbf{J} \cdot n \, d\mathbf{a} \]
\[ \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt} \int_S \mathbf{B} \cdot n \, d\mathbf{a} \]
\[ \int_S \mathbf{B} \cdot n \, d\mathbf{a} = 0 \]

\[ \text{mmf} = \mathbf{N} \mathbf{I} = \Phi \mathbf{R} \]
\[ \mathbf{R} = \frac{\mathbf{L}}{\mu \mathbf{A}} \quad \Phi = \mathbf{B} \mathbf{A} \quad \mathbf{B} = \mu \mathbf{H} \quad \kappa = \mathbf{N} \Phi \]

\[ \omega_{m} = \int_{x=\text{const}} i \, dx \]
\[ \omega'_{m} = \int_{x=\text{const}} i' \, dx \]
\[ \omega_{m} + \omega'_{m} = \mathbf{N} \mathbf{i} \]

\[ f = -\frac{\partial \omega}{\partial x} \]
\[ f = \frac{\partial \omega'}{\partial x} \quad \text{For rotation, } x \rightarrow \mathbf{\Theta} \]
\[ f \rightarrow T \]

\[ \mathbf{E} \mathbf{F} \mathbf{E} = \int_{a}^{b} i \, dx \quad \mathbf{E} \mathbf{F} \mathbf{M} = -\int_{a}^{b} f \, dx \]

\[ \omega_{mb} - \omega_{na} = \mathbf{E} \mathbf{F} \mathbf{E} + \mathbf{E} \mathbf{F} \mathbf{M} \]
\[ x(t+\Delta t) \approx x(t) + \left. \frac{dx}{dt} \right|_{t} \Delta t \]

\[ t_0 \]
Problem 1 (25 pts.)

For the structure drawn below, the movable member is constrained to move left and right only as indicated in the figure where “x” is the distance to the right edge of the movable member. The large member with the coil is fixed, and the depth into the page for both members is 2cm. The gap g is 1mm, and the number of turns $N = 100$. Find:

a. Total reluctance of the magnetic circuit.
b. Flux linkage, $\lambda$. (defined for the voltage polarity shown)
c. Co-energy, $W_m$.
d. Force of electrical origin, $f_e$.
e. An expression for the voltage, $v$.

Express all of these as functions of current and/or position and/or velocity and/or time as appropriate. You may neglect fringing in the gap, and you may assume the iron is infinitely permeable.

\[ R = \frac{2g}{\mu_0 l d x} = \frac{0.062}{\frac{4\pi \times 10^{-7} \times 0.02}{x}} \]

\[ = \frac{7.96 \times 10^4}{x} \]

b) $\phi R = 100i$

\[ \phi = \frac{100i x}{7.96 \times 10^4} \]

\[ \lambda = \frac{i x}{7.96} \]

c) $\omega_m = \frac{x i^2}{15.92}$

d) $f_e = \frac{i^2}{15.92}$

e) $v = iR + \frac{x}{7.96} \frac{di}{dt} + \frac{i}{7.96} \frac{dx}{dt}$
Problem 2 (25 pts)

A single-phase, four-pole machine is shown below. There are only two sources involved - the stator current goes into $s_1$, then out $s_2$, ($N_s$ times), then into $s_3$, and out $s_4$ ($N_s$ times). The stator voltage terminals are $s_1$ (+) and $s_4$ (-). The same notation is used for the rotor coil where there are $N_r$ turns for each part of the rotor coil. The air gap is uniform with distance $g$. The iron is infinitely permeable. The mean radius is $r$, and depth into the paper is $\ell$.

![Diagram of a single-phase, four-pole machine](image)

a) Assign the 8 different magnetic field intensity vectors $H_1$ to $H_8$ and write the eight equations that you would need to compute each $H$ in terms of the currents $i_s$ and $i_r$.

b) Write the equation you need to compute the stator flux linkage in terms of the different $H$ quantities and angle theta. The stator terminal voltage (with the polarity given) is equal to the time derivative of this stator flux linkage.

\[
\begin{align*}
H_1 - H_9 &= N_s i_s \\
H_2 - H_9 &= N_s i_j - N_s i_r \\
H_3 - H_9 &= -N_s i_r \\
H_4 - H_9 &= 0
\end{align*}
\]

\[
\begin{align*}
M_0 H_1 \sigma r e + M_0 H_2 \left( \frac{\pi}{2} - \theta \right) r e + M_0 H_3 \sigma r e + M_0 H_4 \left( \frac{\pi}{2} - \theta \right) r e \\
+ M_0 H_5 \sigma r e + M_0 H_6 \left( \frac{\pi}{2} - \theta \right) r e + M_0 H_7 \sigma r e + M_0 H_8 \left( \frac{\pi}{2} - \theta \right) r e \\
= 0
\end{align*}
\]

\[
\begin{align*}
\phi_s &= N_s M_0 H_1 \sigma r e + N_s M_0 H_2 \left( \frac{\pi}{2} - \theta \right) r e + N_s M_0 H_3 \sigma r e + N_s M_0 H_4 \left( \frac{\pi}{2} - \theta \right) r e
\end{align*}
\]
Problem 3 (25 pts.)

Given the electromechanical relay shown below with the typical parameters as indicated. The rectangular pieces are iron with infinite permeability and cross sectional area $A$. The spring zero-force distance (script $f$) is calibrated to the distance $x$.

![Electromechanical Relay Diagram]

a) Find an expression for the force of electrical origin acting on the mass $M$ in terms of the various parameters in the figure and the coil current.

b) Write the complete dynamic model for this device in state space form (3 ordinary differential equations). Assume inputs are voltage and external force. Add a resistor to the coil.

c) If the mass is held fixed at an initial position $x_0$ while the coil is energized to a current value of $i_0$ and a flux linkage value of $\lambda_0$, find an expression for the energy transferred from the electrical system into the coupling field.

d) If the flux linkage is then held constant at $\lambda_0$ while the mass is moved from position $x_0$ to $x_1$, find an expression for the energy transferred from the mechanical system into the coupling field.

\[ a) \quad H_1(1-x) + H_2(1-x) = Ni \]
\[ M_0 H_1 A_1 - M_0 H_2 A_2 = 0 \]
\[ A_1 = A_2 \quad H_1 = H_2 \]

\[ H_1 = \frac{Ni}{2(1-x)} \quad B_1 = \frac{M_0 Ni}{2(1-x)} \quad \phi_{up} = \frac{M_0 A N^2 i^2}{2(1-x)} \]

\[ \omega_m = \frac{M_0 A N^2 i^2}{4(1-x)} \quad f = \frac{M_0 A N^2 i^2}{4(1-x)^2} \]

\[ b) \quad \frac{dx}{dt} = \nu \quad \frac{d\nu}{dt} = m \left[ f_{ext} + \frac{M_0 A N^2 i^2}{4(1-x)^2} - k(x-x_0) - 0 \nu \right] \]

\[ \nu = iR + \frac{M_0 A N^2}{2(1-x)} \frac{di}{dt} + \frac{M_0 A N^2 \nu}{2(1-x)^2} \]
\[ \frac{di}{dt} = \frac{2(1-x)}{M_0 A N^2} \left[ \nu - iR - \frac{M_0 A N^2 \nu}{2(1-x)^2} \right] \]
\[ C) \quad EFE = \int_{a-b}^{a+b} c \, dx = \int_{0}^{d} \frac{2(d-x_0)}{\mu_0 AN^2} \, dx = \frac{(d-x_0)^2}{\mu_0 AN^2} \]

\[ d) \quad EFM = -\int_{b-c}^{b+c} f \, dx = -\int_{x_0}^{x_1} \frac{\mu_0 AN^2 \cdot 2}{\sqrt{(d-x)^2}} \, dx \]
\[ = -\int_{x_0}^{x_1} \frac{\mu_0 AN^2}{\sqrt{4(d-x)^2}} \left( \frac{2(d-x)}{\mu_0 AN^2} \right)^2 \, dx \]
\[ = -\int_{x_0}^{x_1} \frac{\gamma_0^2}{\mu_0 AN^2} \, dx = \frac{\gamma_0^2 (x_0 - x_1)}{\mu_0 AN^2} \]
Problem 4 (25 pts.)

In the following state-space model the initial conditions are $X_1(0) = 1$, $X_2(0) = 0.5$, and $X_3(0) = 5$.

\[
\begin{align*}
    \dot{X}_1 &= X_2 \\
    \dot{X}_2 &= -0.1X_2X_3 - X_1^3 + X_1X_3 \\
    \dot{X}_3 &= -2X_2X_3 - X_3 + 4
\end{align*}
\]

a) With a time step of 0.1 seconds, use Euler’s method to find $X_1$, $X_2$, and $X_3$ at $t=0.1$ seconds and 0.2 seconds.

b) Find all possible static equilibrium points.

\[a) \] \[X_1(0.1) = 1 + 0.5x_{1.1} = 1.05 \]
\[X_2(0.1) = 0.5 + \left[ -0.1 \times 0.5 \times 5 \right] \times 1 = 0.875 \]
\[X_3(0.1) = 5 + \left[ -2 \times 0.5 \times 5 - 5 + 4 \right] \times 1 = 4.4 \]

\[X_1(0.2) = 1.05 + 0.8 \times 5 \times 0.1 = 1.1375 \]
\[X_2(0.2) = 0.875 + \left[ -0.1 \times 0.875 \times 4.4 - 1.05^2 + 1.05 \times 4.4 \right] \times 1 = 1.183 \]
\[X_3(0.2) = 4.4 + \left[ -2 \times 0.875 \times 4.4 - 4.4 + 4 \right] \times 1 = 3.59 \]

\[b) \] \[x_2^e = 0 \quad x_3^e = 4 \]
\[x_1^e = 0 \quad \text{or} \quad x_1^e = \pm 2 \quad \text{or} \quad x_1^e = -2 \]