Chapter 5

Stability

Linear Systems

#28 Review

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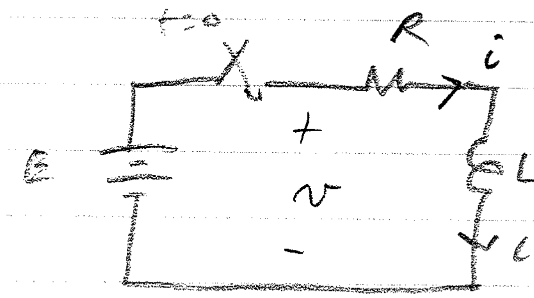
Review of Math 285 (& ECE 210)

1st Order




$$v = iR + L\frac{di}{dt}$$

Put in state space form

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}v$$

A

$i(0) = 0$

$v = 0 \quad t < 0$

$v = E \quad t \geq 0$

Solution is:

$$i = ae^{\lambda t} + i_{ss}$$

DC steady state (Equilibrium) $\quad 0 = -\frac{R}{L}i + \frac{1}{L}E$

$$\boxed{i_{ss} = \frac{E}{R}}$$

Now find $\lambda$ and $a$

$$|A - \lambda I| = |\lambda I - A| = \left|\lambda + \frac{R}{L}\right| = \lambda + \frac{R}{L}$$

characteristic equation $\quad \lambda + \frac{R}{L} = 0$

$$\lambda = -\frac{R}{L} \quad \text{(Eigenvalue)}$$

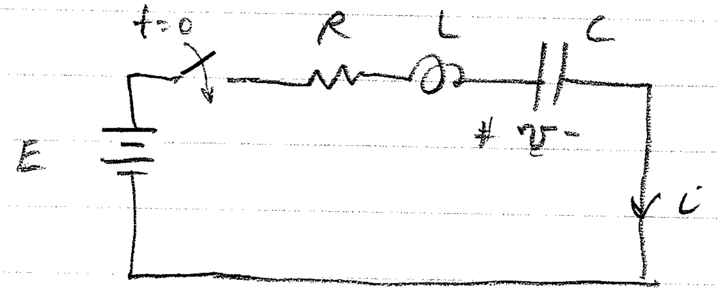
$$i = a e^{-\frac{R}{L}t} + \frac{E}{R}$$

Find $a$ using initial condition

$$i(0) = 0 = a + \frac{E}{R} \longrightarrow \boxed{a = -\frac{E}{R}}$$ Like an (Eigenvector)

$$i = -\frac{E}{R} e^{-\frac{R}{L}t} + \frac{E}{R}$$ check $t=0$ & $t=\infty$

2nd order




Before switch is closed, $i = 0$ $\quad v = v^0$

After switch is closed,

$$E = iR + L\frac{di}{dt} + v$$

$$i = C\frac{dv}{dt}$$

Put in state space form (no x's though)

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}v + \frac{1}{L}E \qquad i(0) = 0$$

$$\frac{dv}{dt} = \frac{1}{C}i \qquad v(0) = v^0$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [E] \qquad \begin{matrix} i(0) = 0 \\ v(0) = v_0 \end{matrix}$$

Solve this for $i(t)$ and $v(t)$

Solution will be of the form:

$$\left.\begin{aligned} i &= a e^{\lambda_1 t} + b e^{\lambda_2 t} + i_{ss} \\ v &= c e^{\lambda_1 t} + d e^{\lambda_2 t} + v_{ss} \end{aligned}\right\}$$

Note: Stable if $\lambda_1 < 0$ $\lambda_2 < 0$

Steady state solution first:

DC inputs, so equilibrium is

$$\left.\begin{aligned} 0 &= -\frac{R}{L} i - \frac{1}{L} v + \frac{1}{L} E \\ 0 &= \frac{1}{C} i \end{aligned}\right\} \qquad \begin{aligned} i_{ss} &= 0 \\ v_{ss} &= E \end{aligned}$$

Now compute $\lambda_1$ $\lambda_2$ (Eigenvalues)

$\begin{pmatrix} a \\ c \end{pmatrix}$ & $\begin{pmatrix} b \\ d \end{pmatrix}$ (like Eigenvectors)

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$$|A - \lambda I| = |\lambda I - A| = \begin{vmatrix} \lambda + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

Characteristic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

Roots $\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$ (Eigenvalues)

Note: Could be complex

Now find $a, b, c, d$ from initial conditions

$t=0$ $\quad i(0) = 0 = a + b + 0$

$v(0) = v^0 = c + d + E$

$t=0$ derivatives

$$\left.\frac{di}{dt}\right|_0 = -\frac{R}{L} i(0) - \frac{1}{L} v(0) + \frac{1}{L} E = a\lambda_1 + b\lambda_2$$

$$\left.\frac{dv}{dt}\right|_0 = \frac{1}{C} i(0) = c\lambda_1 + d\lambda_2$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{E}{L} - \frac{v^0}{L} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} v^0 - E \\ 0 \end{bmatrix}$$

Solve these 2×2's for $\begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} c \\ d \end{pmatrix}$ (like eigenvectors)

Chapter 5

Stability

Linearization about an equilibrium point

<u>General:</u>

$$\dot{x} = f(x, u)$$

Equilibrium ($u$ constant)

$0 = f(x^e, u)$ Solve for $x^e$

Linearize around $x^e$

Taylor series in $x$

$$f(x,u) = \underbrace{f(x^e, u)}_{\text{zero}} + \left.\frac{df}{dx}\right|_{x=x^e}(x - x^e) + \underbrace{\frac{1}{2}\frac{d^2f}{dx^2}(x-x^e)^2 + \cdots}_{\text{h.o.t.}}$$

$$\Delta x \stackrel{\Delta}{=} x - x^e$$

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"Linearized" dynamic model

$$\Delta\dot{x} = \frac{d}{dt}(x - x^e) = \frac{dx}{dt} = \dot{x} = f(x,u)$$

$$\Delta\dot{x} = \left.\frac{df}{dx}\right|_{x=x^e}\Delta x + \frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x^e}\Delta x^2 + \cdots$$

$$\Delta\dot{x} \approx \left.\frac{df}{dx}\right|_{x=x^e}\Delta x \qquad \text{linear system}$$

Extend to higher order systems

Example

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -6\sin x_1 - 4x_2 + 3$$

Equilibrium $\quad 0 = x_2^e \qquad x_2^e = 0$

$$0 = -6\sin x_1^e + 3 \qquad x_1^e = \sin^{-1} 0.5$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$$

Linearize

$$\Delta \dot{x}_1 = \Delta x_2$$

$$\Delta \dot{x}_2 = (-6\cos x^e)\Delta x_1 - 4\,\Delta x_2$$

$$A = \begin{bmatrix} 0 & 1 \\ -6\cos x^e & -4 \end{bmatrix}$$

Compute eigenvalues

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 6\cos x^e & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 6\cos x^e$$

Characteristic equation

$$\lambda^2 + 4\lambda + 6\cos x^e = 0$$

$$\lambda = -\frac{4}{2} \pm \frac{1}{2}\sqrt{(4)^2 - 24\cos x^e}$$

Check stability of the two equilibrium points

$$\begin{array}{r} 1.732 \\ 12 \\ \hline 3464 \\ 1732 \\ \hline 20.784 \approx 21 \end{array}$$

$x_1^e = \pi/6 \qquad x_2^e = 0$

$$\lambda = -2 \pm \frac{1}{2}\sqrt{16 - 24\cos\pi/6}$$

$$= -2 \pm j\frac{\sqrt{5}}{2} \qquad \text{Stable}$$

$x_1^e = 5\pi/6 \qquad x_2^e = 0$

$$\lambda = -2 \pm \frac{1}{2}\sqrt{16 - 24\cos\frac{5\pi}{6}}$$

$$= -2 \pm \frac{\sqrt{38}}{2} \qquad \text{unstable}$$

Chapter 6

Synchronous machines

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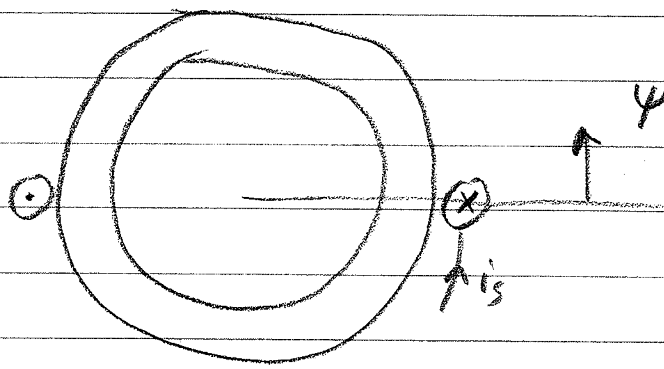
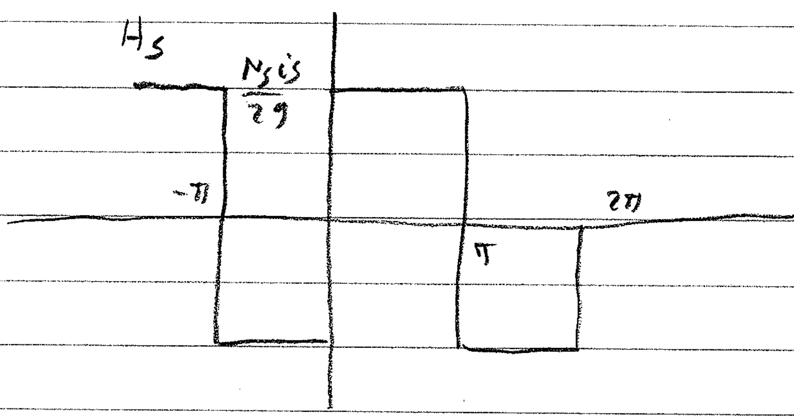
## Pulsating magnetic field (Single phase)




$$H_s \approx \frac{N_s i_s}{2g} \sin\psi$$

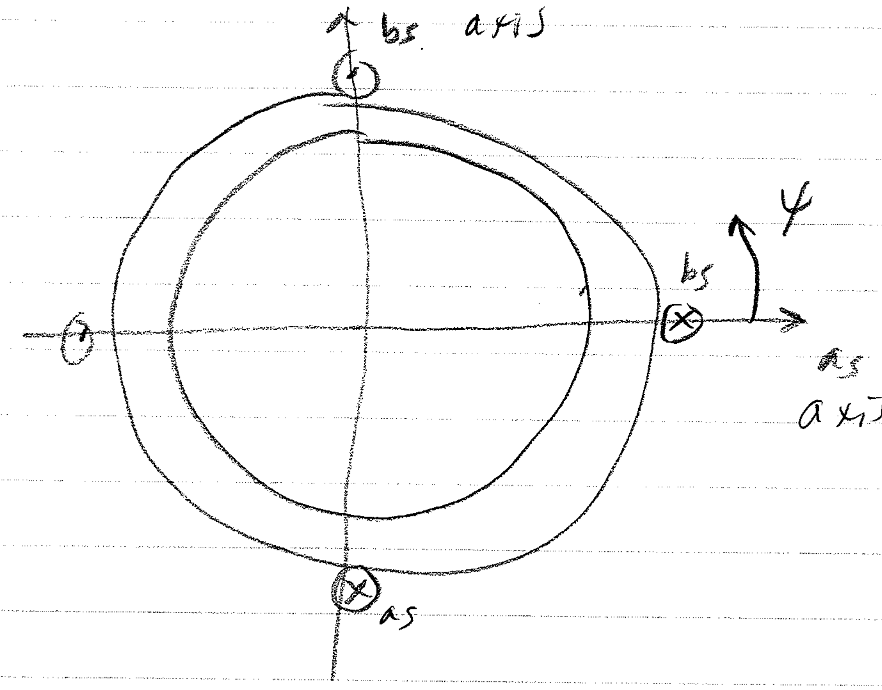
For $i_s = I_s \cos w_s t$

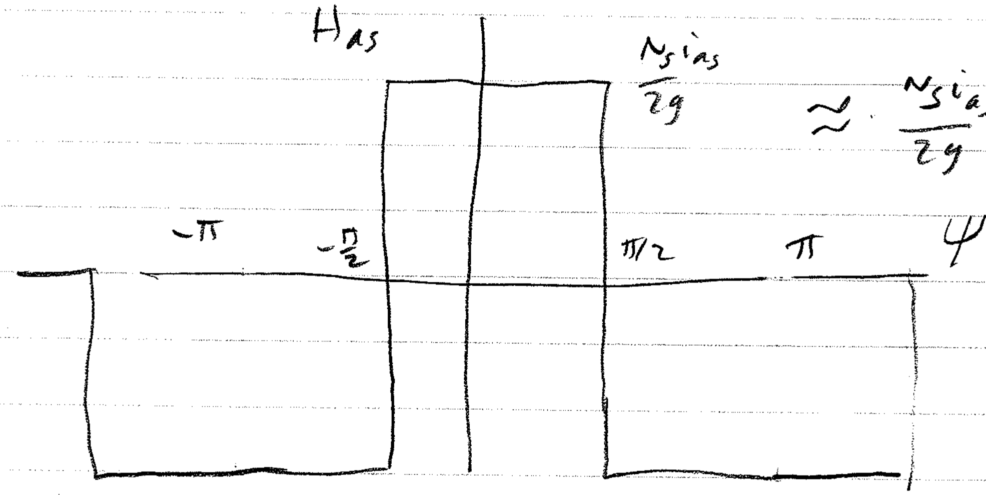
$$H_s = \frac{N_s I_s}{2g} \cos w_s t \sin\psi$$

At $t=0$ + max at $\psi = \pi/2$

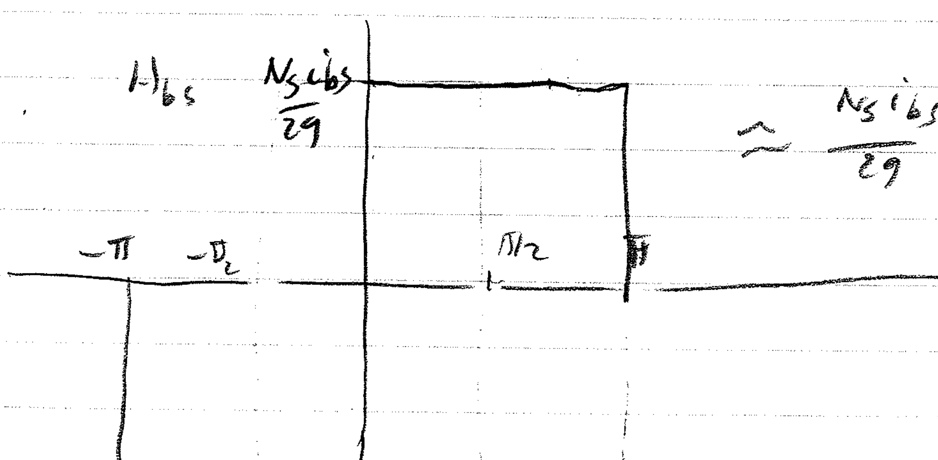
At $t=\epsilon$ + max still at $\psi = \pi/2$

## Revolving magnetic field (two phase)




$$\approx \frac{N_s i_{as}}{2g} \cos\psi$$




$$\approx \frac{N_s i_{bs}}{2g} \sin\psi$$

$$H_s = H_{as} + H_{bs}$$

$$= \frac{N_s}{2g} i_{as} \cos\psi + \frac{N_s}{2g} i_{bs} \sin\psi$$

Assume: $i_{as} = I_s \cos w_s t$

$i_{bs} = I_s \sin w_s t$

$$H_s = \frac{N_s I_s}{2g}\left[\cos\psi \cos w_s t + \sin\psi \sin w_s t\right]$$

$$= \frac{N_s I_s}{2g} \cos(w_s t - \psi)$$

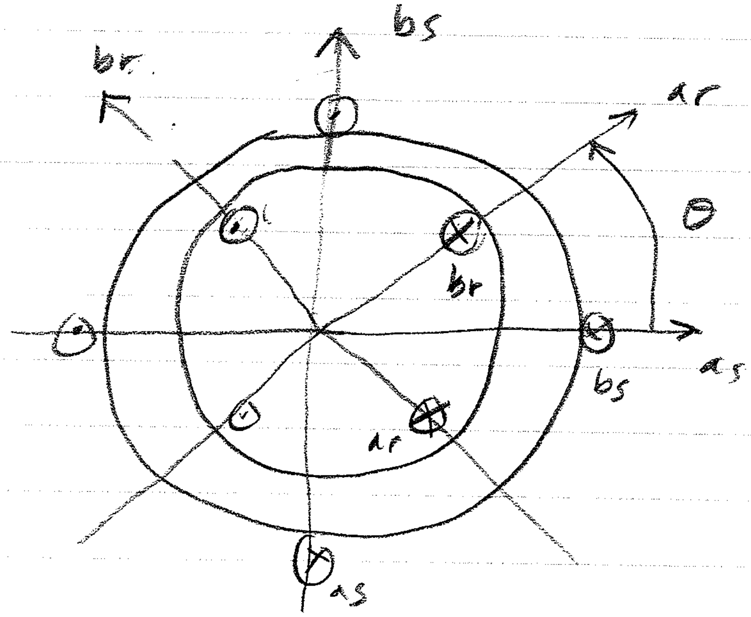
Revolving magnetic field

i.e. at $t = 0$ + peak is at $\psi = 0$

$t = \epsilon$ + peak is at $\psi = w_s \epsilon$

Revolves Counter Clockwise

# Two Phase




$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{bmatrix} = \begin{bmatrix} L_s & 0 & m\cos\theta & -m\sin\theta \\ 0 & L_s & m\sin\theta & m\cos\theta \\ m\cos\theta & m\sin\theta & L_r & 0 \\ -m\sin\theta & m\cos\theta & 0 & L_r \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{bmatrix}$$

Stor 05

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$$W_m' = \left(\frac{1}{2} L_s i_{as}^2\right) + \left(0 + \frac{1}{2} L_s i_{bs}^2\right)$$

$$+ \left(m\cos\theta\, i_{as} i_{ar} + m\sin\theta\, i_{bs} i_{ar} + \frac{1}{2} L_r i_{ar}^2\right)$$

$$+ \left(-m\sin\theta\, i_{as} i_{br} + m\cos\theta\, i_{bs} i_{br} + 0 + \frac{1}{2} L_r i_{br}^2\right)$$

$$T^e = -m\sin\theta\, i_{as} i_{ar} + m\cos\theta\, i_{bs} i_{ar}$$

$$- m\cos\theta\, i_{as} i_{br} - m\sin\theta\, i_{bs} i_{br}$$

Now assume steady state again:

$i_{as} = I_s \cos w_s t \qquad i_{bs} = I_s \sin w_s t$

$i_{ar} = I_r \cos w_r t \qquad i_{br} = I_r \sin w_r t$

$\theta = w_m t + \gamma$

$$T^e = -M I_s I_r \Big[ \sin(w_m t + \gamma) \cos w_s t \cos w_r t$$
$$- \cos(w_m t + \gamma) \sin w_s t \cos w_r t$$
$$+ \cos(w_m t + \gamma) \cos w_s t \sin w_r t$$
$$+ \sin(w_m t + \gamma) \sin w_s t \sin w_r t \Big]$$

$$T^e = -mI_s I_r \sin(w_m t + \gamma)\left[\cos(w_s - w_r)t\right]$$

$$+ mI_s I_r \cos(w_m t + \gamma)\left[\sin(w_s - w_r)t\right]$$

$$= -mI_s I_r \sin\left((w_m - w_s + w_r)t + \gamma\right)$$

Assume: $w_m = w_s - w_r$

$$T^e = -mI_s I_r \sin\gamma \qquad \text{(constant torque!)}$$

This frequency condition guarantees constant torque which is consistent with the constant speed assumption.

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Plus, look at voltages with $\theta = w_m t + \gamma$

$$v_{as} = I_s \cos w_s t \, R_s + \frac{d}{dt}\Big[ L_s I_s \cos w_s t + m \cos(w_m t + \gamma)\, i_{ar} - m \sin\theta \, i_{br} \Big]$$

Let $i_{ar} = I_r \cos w_r t$

$i_{br} = I_r \sin w_r t$

$$v_{as} = R_s I_s \cos w_s t - w_s L_s I_s \sin w_s t + \frac{d}{dt}\Big[ m \cos(w_m t + \gamma) I_r \cos w_r t - m \sin(w_m t + \gamma) I_r \sin w_r t \Big]$$

$$= R_s I_s \cos w_s t - w_s L_s I_s \sin w_s t + \frac{d}{dt}\Big[ m I_r \cos\big((w_m + w_r)t + \gamma\big)\Big]$$

$$= R_s I_s \cos w_s t - w_s L_s I_s \sin w_s t - (w_m + w_r) m I_r \sin\big((w_m + w_r)t + \gamma\big)$$

Require $w_m + w_r = w_s$

$$v_s = R_s I_s \cos w_s t - w_s L_s I_s \sin w_s t - w_s m I_r \sin(w_s t + \gamma)$$

Sum of Sinusoids of one frequency is sinusoidal same frequency

Note: not same as rotor position! — Bad book notation

$$V_s \cos(\omega_s t + \theta) = R_s I_s \cos(\omega_s t) - \omega_s L_s I_s \sin \omega_s t$$

$$- \omega_s M I_r \sin(\omega_s t + \gamma)$$

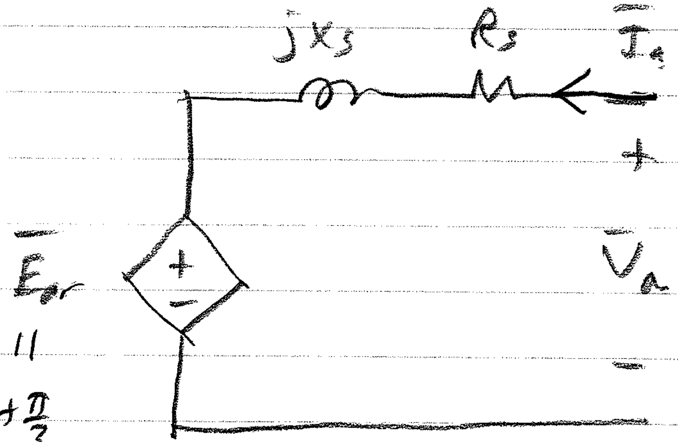
Stop here, break 2005

Phasor domain:

$$\frac{V_s}{\sqrt{2}} \angle \theta = \frac{R_s I_s}{\sqrt{2}} \angle 0 + j\omega_s L_s \frac{I_s}{\sqrt{2}} \angle 0 + j \frac{\omega_s M I_r}{\sqrt{2}} \angle \gamma$$

use $\omega_m = \omega_s$ $\quad \omega_r = 0$ ($i_{ar}$ = dc current)

Synchronous machine

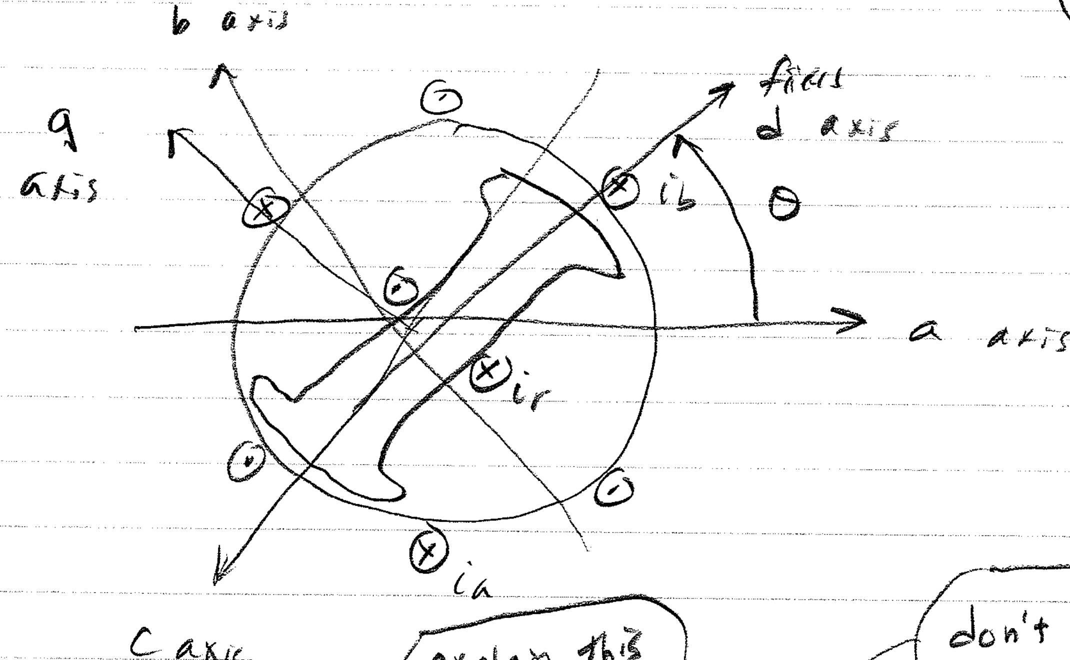



$$\bar{E}_{ar} = \frac{\omega_s M I_r}{\sqrt{2}} \angle \gamma + \frac{\pi}{2}$$

2006 No $\frac{\pi}{2}$ $\quad$ Can show $v_{ar} = V_r$ (dc)

# 3-phase Synchronous machine

Do this at end



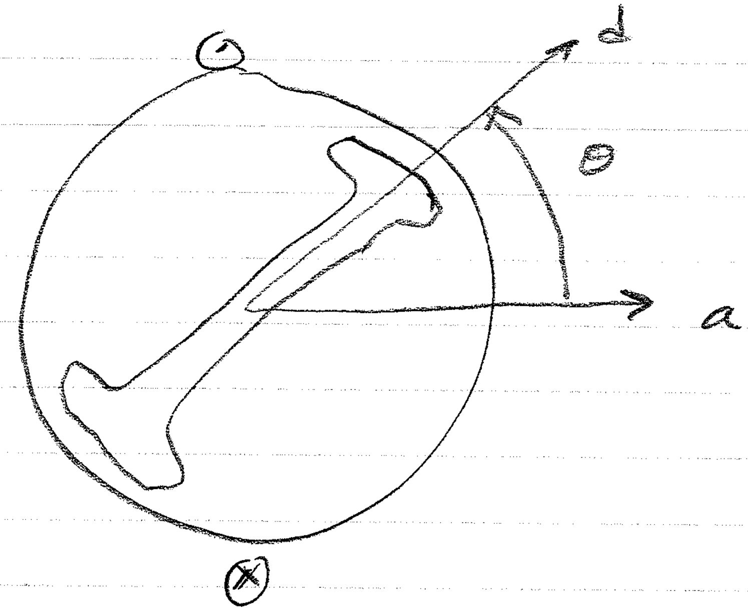

explain this (see next page)

don't try to explain these terms

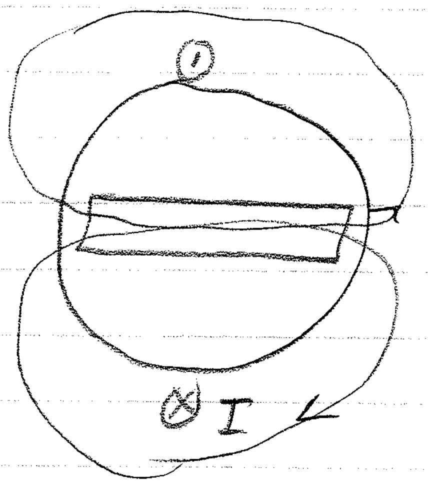
$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_r \end{bmatrix} = \begin{bmatrix} L_0 + L_2\cos 2\theta & -M_0 + M_2\cos 2(\theta - 60) & -M_0 + M_2\cos 2(\theta + 60) & M\cos\theta \\ -M_0 + M_2\cos 2(\theta - 60) & L_0 + L_2\cos 2(\theta - 120^\circ) & -M_0 + M_2\cos 2(\theta - 180^\circ) & M\cos(\theta - 120^\circ) \\ -M_0 + M_2\cos 2(\theta + 60) & -M_0 + M_2\cos 2(\theta - 180^\circ) & L_0 + L_2\cos 2(\theta + 120^\circ) & M\cos(\theta + 120^\circ) \\ M\cos\theta & M\cos(\theta - 120) & M\cos(\theta + 120^\circ) & L_r \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_r \end{bmatrix}$$

explain this column

# Salient Pole



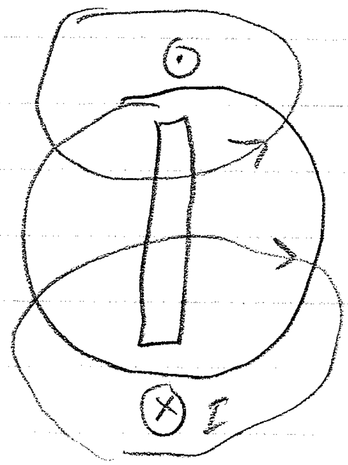

$\theta = 0$




max flux linkage

$$\lambda = L_{max} I \quad \left(\propto \frac{1}{g_{small}}\right)$$

$\theta = \pi/2$




Minimum flux linkage

$$\lambda = L_{min} I \quad \left(\propto \frac{1}{g_{big}}\right)$$

$\theta = \pi$ Same as $\theta = 0$

$\theta = \frac{3\pi}{2}$ Same as $\theta = \pi/2$

Propose

$$\lambda = (L_0 + L_2 \cos 2\theta) I$$

$\theta = 0$ $L_0 + L_2 = L_{max}$

$\theta = \pi/2$ $L_0 - L_2 = L_{min}$

$$L_0 = \frac{L_{max} + L_{min}}{2}$$

$$L_2 = \frac{L_{max} - L_{min}}{2}$$

Round Rotor Case $L_{max} = L_{min}$ $(L_2 = 0)$

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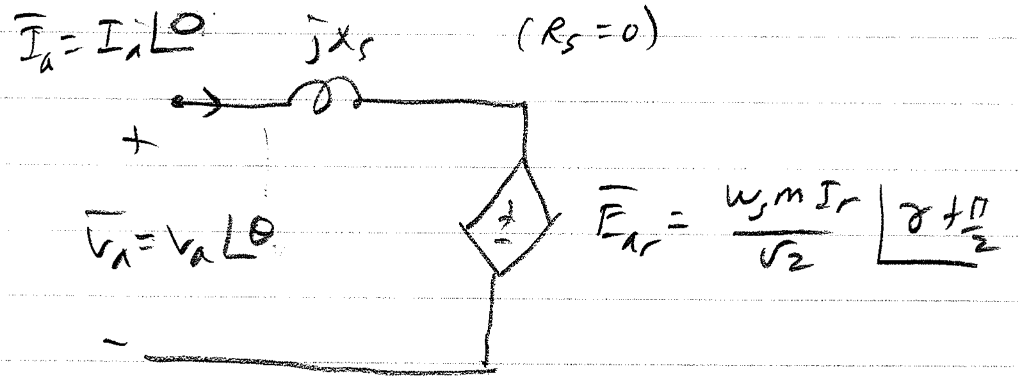
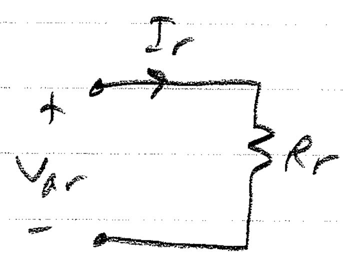
$$w_m' = \int \lambda \, di = \text{See book}$$

$$T^e = \frac{\partial w_m'}{\partial \theta} = \text{See book}$$

Same result as 2-phase




Note: Book uses $\theta$ – this is not same $\theta$ as rotor position

Book selected $i_{as}$ as phase angle zero,

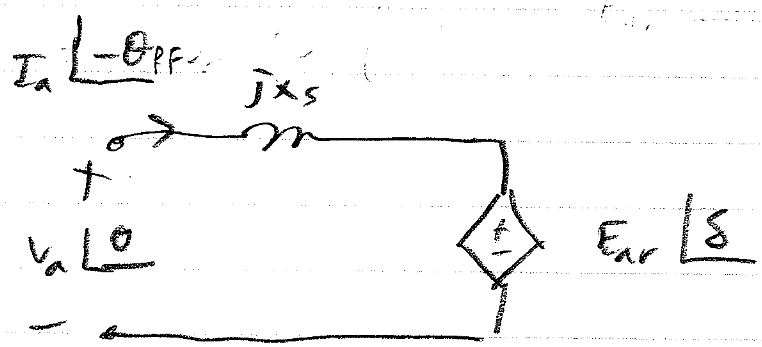
so: $\theta = \text{pf angle } (\angle \bar{V}_a - \angle \bar{I}_a)$

$$V_a \angle \theta = I_a \angle 0 \; jX_s + \frac{\omega_m M I_r}{\sqrt{2}} \angle \gamma + \frac{\pi}{2}$$

multiply through by $e^{-j\theta}$

$$V_a\angle 0 = I_a\angle{-\theta_{pf}}\, jX_s + \frac{\omega_s m I_r}{\sqrt{2}}\angle\gamma + \frac{\pi}{2} - \theta_{pf}$$

$$\delta \triangleq \gamma + \frac{\pi}{2} - \theta_{pf} \quad \text{(Torque angle)}$$




$$\overline{S}_{IN,3\phi} = 3\overline{V}_a\overline{I}_a^* = 3V_a I_a\angle\theta_{pf}$$

$$= 3V_a\angle 0\left(\frac{V_a\angle 0 - E_{ar}\angle\delta}{jX_s}\right)^*$$

$$= \frac{3V_a^2}{X_s}\angle 90^\circ - \frac{3V_a E_{ar}}{X_s}\angle 90^\circ - \delta$$

$$P_{IN_{3\phi}} = -\frac{3V_a E_{ar}}{X_s}\cos(90° - \delta) = -\frac{3V_a E_{ar}}{X_s}\sin\delta$$

$$T^e = \frac{P_{IN_{3\phi}}}{\omega_s} = -\frac{3V_a E_{ar}}{\omega_s X_s}\sin\delta$$ (Error in book)

Eq. 6.60

$\delta < 0$ motor

$\delta > 0$ Generator

$$Q_{IN_{3\phi}} = \frac{3V_a^2}{X_s} - \frac{3V_a E_{ar}}{X_s}\cos\delta$$

$Q_{IN_{3\phi}} < 0$ Overexcited ($E_{ar}\cos\delta > V_a$)

$Q_{IN_{3\phi}} > 0$ Underexcited ($E_{ar}\cos\delta < V_a$)

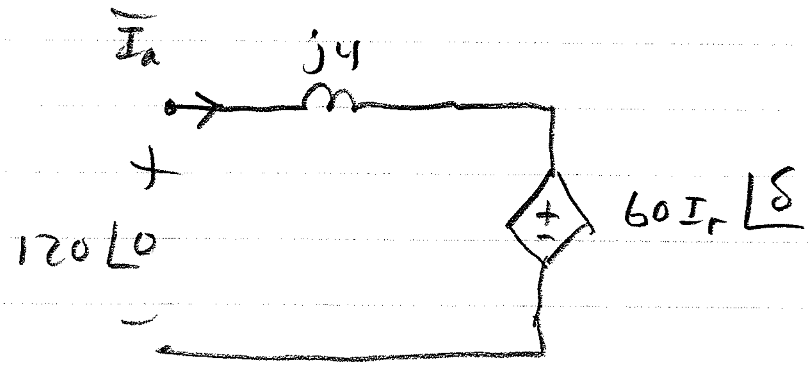
Note: $Q_{IN_{3\phi}}$ is determined by $E_{ar}$ (Excitation)

Example

$$\bar{V}_a = 120\angle 0$$

$$X_s = 4\,\Omega$$

$$\frac{\omega_s m}{\sqrt{2}} = 60\,\Omega$$



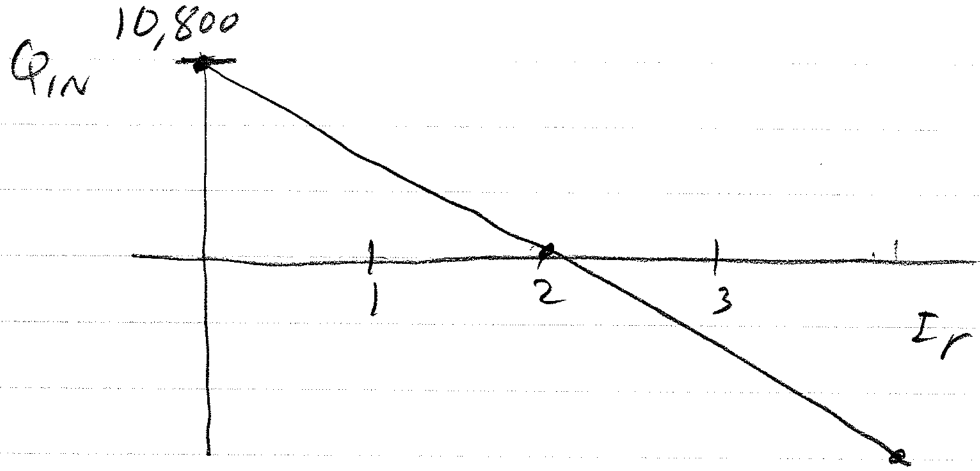

$$P_{IN\,3\phi} = -\frac{3 \times 120 \times 60 I_r}{4} \sin\delta = -5400\, I_r \sin\delta$$

$$Q_{IN\,3\phi} = \frac{3 \times 120^2}{4} - \frac{3 \times 120 \times 60 I_r}{4} \cos\delta = 10{,}800 - 5400\, I_r \cos\delta$$

Look at $P_{IN} = 0$ (Synchronous Condenser)

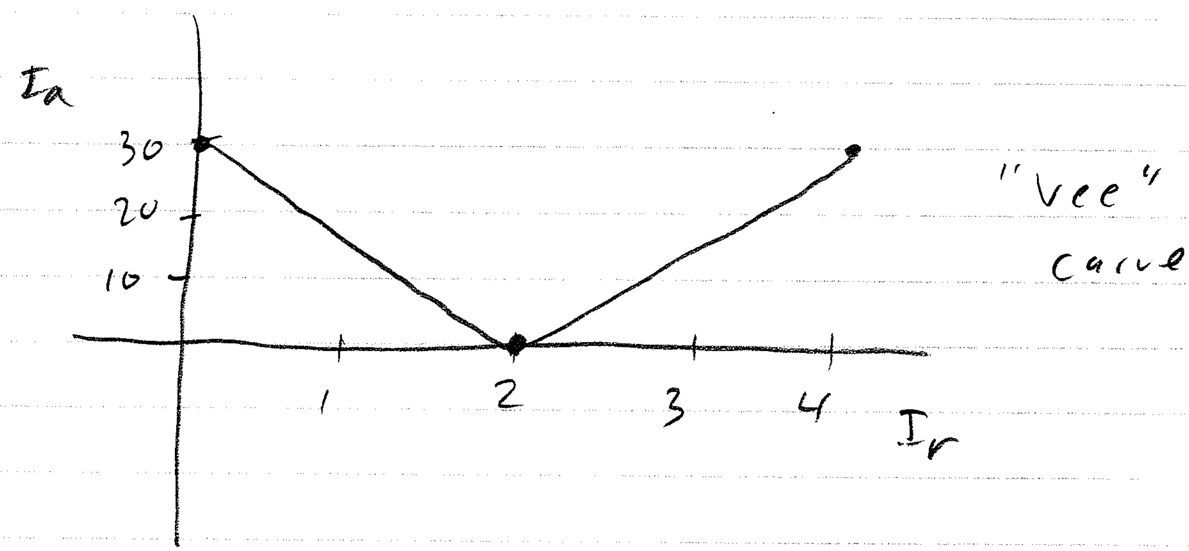
$$\delta = 0$$

$$Q_{IN} = 10{,}800 - 5400\, I_r$$




$$\bar{I}_a = \frac{120\angle 0 - 60 I_r \angle \delta}{j4} = 30\angle -90 - 15 I_r \angle \delta - 90^\circ$$

For $\delta = 0$ $\qquad \bar{I}_a = 0 + j\,(15 I_r - 30)$



For $P_{IN\,3\phi} = 7{,}000$ watts

$$3 \times 120\angle 0 \times \bar{I}_a^{*} = 7{,}000 + j\left(10{,}800 - 5400 I_r \cos\delta\right)$$

$$7000 = -5400\, I_r \sin\delta$$

$$\bar{I}_a = \frac{7{,}000}{360} - j\left(\frac{10{,}800}{360} - \frac{5400}{360} I_r \cos\delta\right)$$

$$7{,}000 = -5400\, I_r \sin\delta$$

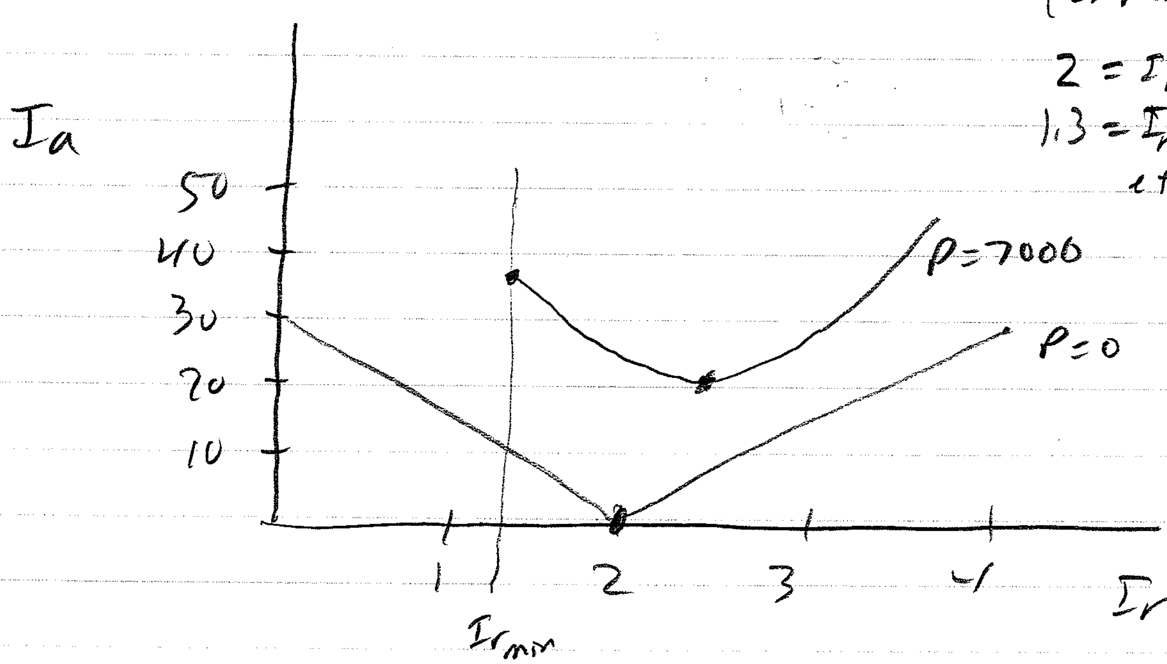
$$I_{r\,min} = \frac{7{,}000}{5400} = 1.3 \text{ Amps} \Rightarrow \delta = 90^\circ$$

$$\left(\bar{I}_a = \frac{7000 + j10{,}800}{360} \approx 20 + j30 \approx 36\angle\right)$$

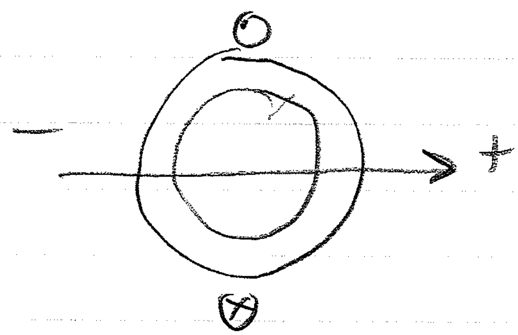
$$I_{a\,min} = \text{Unity pf} = \frac{7000}{360} \approx 20 \text{ Amps} \quad \left(I_r = \sqrt{5.7}\right)$$

(explain this)

$2 = I_r \cos\delta$
$1.3 = I_r \sin\delta$
etc.

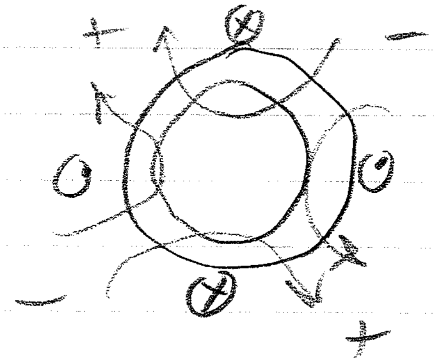


# Poles



2 pole

$1\phi$



4 pole

$1\phi$

$$RPM_{sync} = \frac{120 f}{P}$$

| $P$ | $RPM_{sync}$ |
|---|---|
| 2 | 3600 |
| 4 | 1800 |
| 6 | 1200 |

etc.

$$\theta_{elect} = \frac{P}{2}\,\theta_{mech}$$

$$\omega_{elect} = \frac{P}{2}\,\omega_{mech}$$

$$T^e = \frac{P_{IN3\phi}}{\omega_{mech}}$$

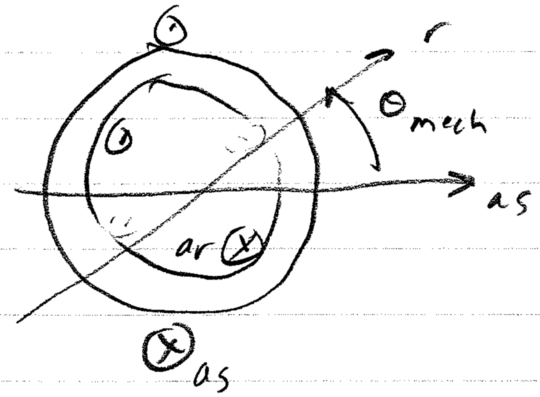
$$= \frac{P_{IN3\phi}}{\frac{2}{P}\omega_s}$$

For 60 Hz $\boxed{\omega_{mech} = \frac{2\pi 60}{P/2}}$

$\omega_s = 2\pi 60$

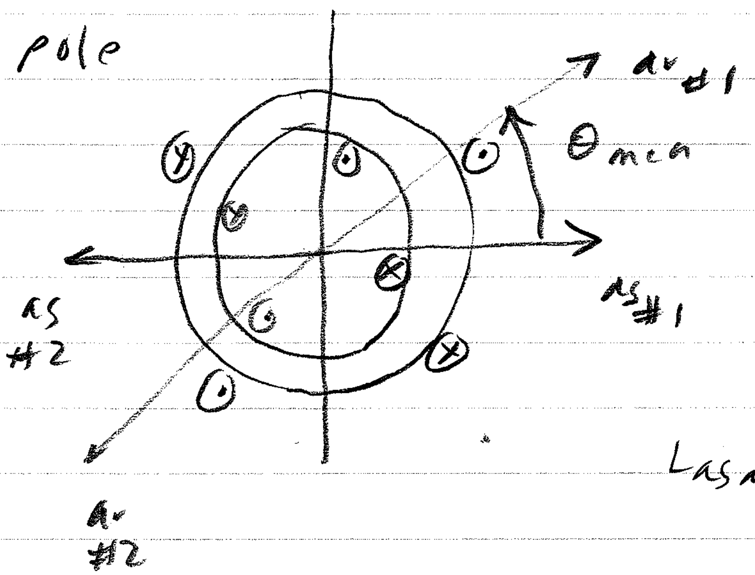
# Mutual Inductance

2 pole




$$L_{as\,ar} = m\cos\theta_{mech}$$

4 pole




$$L_{as\,ar} = m\cos 2\theta_{mech}$$

$$= m\cos\theta_{Elect}$$

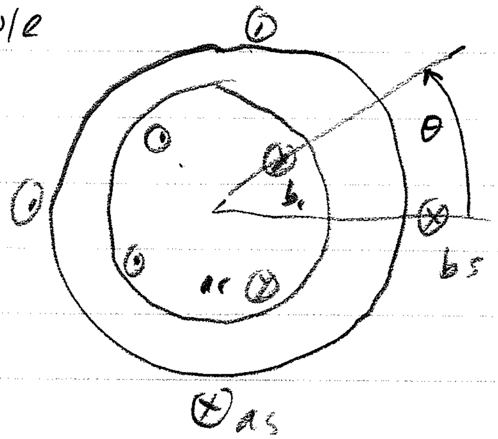
$$\theta_{Elect} = \frac{P}{2}\theta_{mech}$$

## Chapter 7

### Induction machines

Go back to 2-phase

2-pole




$$L = \begin{bmatrix} L_s & 0 & m\cos\theta & -m\sin\theta \\ 0 & L_s & m\sin\theta & m\cos\theta \\ m\cos\theta & m\sin\theta & L_r & 0 \\ -m\sin\theta & m\cos\theta & 0 & L_r \end{bmatrix}$$

$$\theta = \omega_m t + \gamma$$

Earlier result require $\omega_m = \omega_s - \omega_r$

Instead of $\omega_m = \omega_s$ $\omega_r = 0$ (synchronous)

use $\omega_m \neq \omega_s$, so require

$$\omega_r = \omega_s - \omega_m$$

define slip $S \overset{\Delta}{=} \dfrac{\omega_s - \omega_m}{\omega_s}$

so $\boxed{\omega_r = S\omega_s}$ Slip frequency

Assume $i_{ar} = I_{mr}\cos(S\omega_s t + \beta)$ $\quad i_{as} = I_{ms}\cos\omega_s t$

$i_{br} = I_{mr}\sin(S\omega_s t + \beta)$ $\quad i_{bs} = I_{ms}\sin\omega_s t$

$$\theta = \omega_m t + \gamma = (1-S)\omega_s t + \gamma$$

$$v_{as} = R_s I_{ms} \cos w_s t - w_s L_s I_{ms} \sin w_s t +$$

$$\frac{d}{dt}\Big[ m \cos\big((1-s) w_s t + \gamma\big) I_{mr} \cos(s w_s t + \beta)$$

$$- m \sin\big((1-s) w_s t + \gamma\big) I_{mr} \sin(s w_s t + \beta)\Big]$$

$$= R_s I_{ms} \cos w_s t - w_s L_s I_{ms} \sin w_s t +$$

$$\frac{d}{dt}\Big[ m I_{mr} \cos\big((1-s) w_s t + \gamma + s w_s t + \beta\big)\Big]$$

$$= R_s I_{ms} \cos w_s t - w_s L_s I_{ms} \sin w_s t - w_s m I_{mr} \sin(w_s t + \gamma + \beta)$$

$$= V_{ms} \cos(w_s t + \theta_{PF}) \qquad (\text{Recall } \measuredangle i_{as} = 0)$$

$$v_{ar} = R_r I_{mr} \cos(s w_s t + \beta) - s w_s L_r I_{mr} \sin(s w_s t + \beta)$$

$$+ \frac{d}{dt}\Big[ m \cos\big((1-s) w_s t + \gamma\big) I_{ms} \cos w_s t$$

$$+ m \sin\big((1-s) w_s t + \gamma\big) I_{ms} \sin w_s t \Big]$$

$$= R_r I_{mr} \cos(s w_s t + \beta) - s w_s L_r I_{mr} \sin(s w_s t + \beta)$$

$$+ \frac{d}{dt}\Big[ m I_{ms} \cos(s w_s t - \gamma)\Big]$$

$$= R_r I_{mr} \cos(s w_s t + \beta) - s w_s L_r I_{mr} \sin(s w_s t + \beta) - s w_s m I_{ms} \sin(s w_s t - \gamma)$$

$$= V_{mr} \cos(s w_s t + \theta_{vr})$$ NOTE: If $s = 0$ & $V_{mr} = 0$ $I_{mr} = 0$!

Phasors

$$\frac{V_{ms}}{\sqrt{2}}\angle\theta_{vf} = R_s \frac{I_{ms}}{\sqrt{2}}\angle 0 + j\omega_s L_s \frac{I_{ms}}{\sqrt{2}}\angle 0 + j\omega_s m \frac{I_{mr}}{\sqrt{2}}\angle \gamma+\beta$$

$$\frac{V_{mr}}{\sqrt{2}}\angle\theta_{vR} = R_r \frac{I_{mr}}{\sqrt{2}}\angle\beta + jS\omega_s L_r \frac{I_{mr}}{\sqrt{2}}\angle\beta + jS\omega_s m \frac{I_{ms}}{\sqrt{2}}\angle -\gamma$$

OR

$$\frac{V_{mr}}{S\sqrt{2}}\angle\theta_{vr}+\gamma = \frac{R_r}{S}\frac{I_{mr}}{\sqrt{2}}\angle\gamma+\beta + j\omega_s L_r \frac{I_{mr}}{\sqrt{2}}\angle\gamma+\beta + j\omega_s m \frac{I_{ms}}{\sqrt{2}}\angle 0$$

Recall $L_s = L_{ls} + \frac{N_s}{N_R} m$ $\left( m = \frac{N_s N_r \mu_0 \pi r l}{2g} \right)$

$L_r = L_{lr} + \frac{N_r}{N_s} m$

Define $I'_{mr} = I_{mr}\left(\frac{N_r}{N_s}\right)$ $V'_{mr} = V_{mr}\left(\frac{N_s}{N_r}\right)$

$R'_r = R_r\left(\frac{N_s}{N_r}\right)^2$

$L'_{lr} = L_{lr}\left(\frac{N_s}{N_r}\right)^2$

" ' " means "referred" to stator

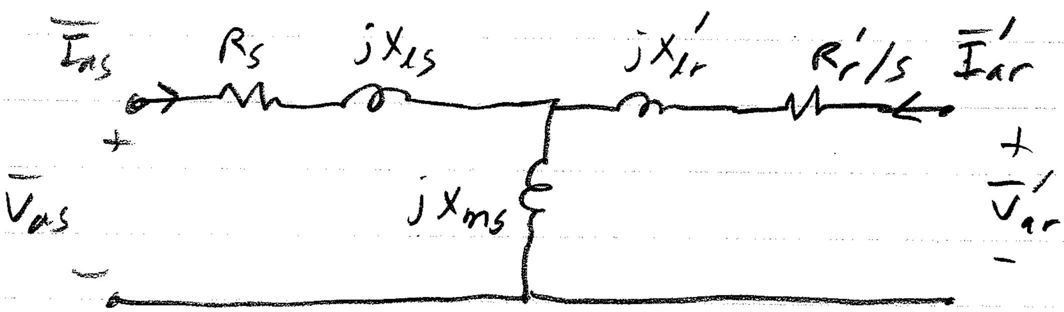
Multiply rotor equation by $\frac{N_s}{N_r}$

$$\frac{V_{ms}}{\sqrt{2}}\angle\theta_{pf} = R_s \frac{I_{ms}}{\sqrt{2}}\angle 0 + j\overbrace{\omega_s L_{ls}}^{X_{ls}} \frac{I_{ms}}{\sqrt{2}}\angle 0 + j\overbrace{\omega_s \frac{N_s}{N_r} m}^{X_{ms}} \frac{I_{ms}}{\sqrt{2}}\angle 0$$

$$+ j\underbrace{\omega_s \frac{N_s}{N_r} m}_{X_{ms}} \frac{I'_{mr}}{\sqrt{2}}\angle \gamma+\beta$$

$$\frac{V'_{mr}}{s\sqrt{2}}\angle\theta_{vr}+\gamma = \frac{R'_r}{s} \frac{I'_{mr}}{\sqrt{2}}\angle\gamma+\beta + j\overbrace{\omega_s L'_{lr}}^{X'_{lr}} \frac{I'_{mr}}{\sqrt{2}}\angle\gamma+\beta$$

$$+ j\underbrace{\omega_s \frac{N_s}{N_r} m}_{X_{ms}} \frac{I'_{mr}}{\sqrt{2}}\angle\gamma+\beta + j\overbrace{\omega_s \frac{N_s}{N_r} m}^{X_{ms}} \frac{I_{ms}}{\sqrt{2}}\angle 0$$




Two types of machines:

a) wound rotor — could connect something at $\bar{V}'_{ar}$

b) squirrel cage $\bar{V}'_{ar} = 0$

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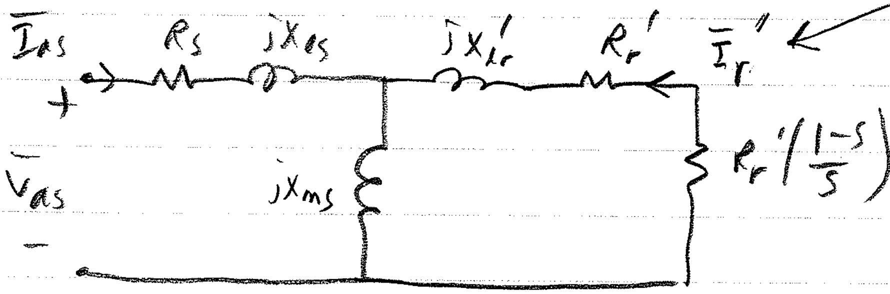
note: $\bar{I}_r' \neq \bar{I}_{ar}'$ angle is different

Let $\bar{V}_{ar}' = 0$

Split $\frac{R_r'}{s}$ into $R_r' + R_r'\left(\frac{1-s}{s}\right)$




(Same ckt as transformer)

Could add $R_c$ for eddy currents + core loss put in parallel with $jX_{ms}$ (like transformer)

– Same circuit for 3 phase machine

Stop 2007

$$P_{IN} = \mathrm{Real}\left(3\,\bar{V}_{as}\bar{I}_{as}^*\right)$$

$$P_{SCL} = 3\,|\bar{I}_{as}|^2 R_s$$

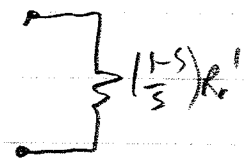
$$P_{RCL} = 3\,|\bar{I}_{ar}'|^2 R_r'$$

$$P_m = 3\,|\bar{I}_{ar}'|^2 R_r'\left(\frac{1-s}{s}\right) \qquad T^e = \frac{P_m}{\omega_m} \text{ (rad/sec)}$$

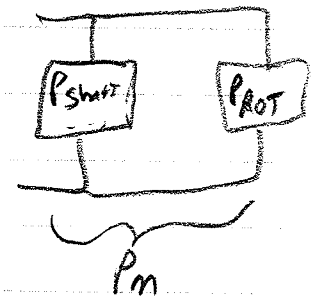
$P_{shaft} = P_m - P_{rot}$ ($P_{rot}$ is in slip)

(output power)

$$P_{AG} = 3|\bar{I}_r'|^2 \frac{R_r'}{s} = \frac{P_m}{(1-s)}$$
(Air gap)




Note: $P_m = (1-s) P_{AG}$ $\Rightarrow$




for $P \neq 2$ $\quad w_m = (1-s)\, w_s \frac{2}{P}$ $\quad (w_s = 2\pi f_s)$
(mr/sec)

$$T^e = \frac{(1-s) P_{AG}}{(1-s) w_s \frac{2}{P}} = \frac{P_{AG}}{\frac{2}{P} w_s}$$

Example 4-pole, 60Hz, 120v (l-n)

$R_s = 0 \quad X_{ls} = 0 \quad X_{ms} = 40\Omega \quad X_{lr}' = 1\Omega \quad R_r' = 1.3\Omega$

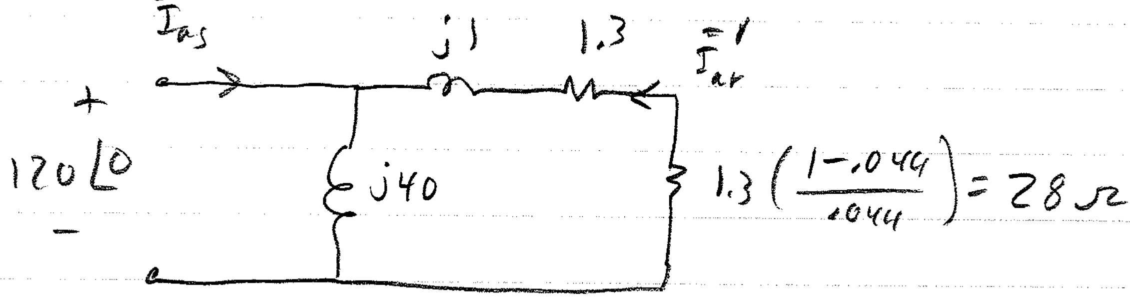
$N_{act} = 1720$ RPm ; neglect $P_c$ and $P_{ROT}$

Find: $s$, $f_r$, $|\bar{I}_{or}'|$, $P_m$, $w_m$, $T^e$, $P_{AG}$, $P_{IN}$, $\bar{I}_{as}$, $Pf$

Soln: $N_s = \frac{120 f_s}{P} = \frac{120 \times 60}{4} = 1800$

$$s = \frac{1800 - 1720}{1800} = 0.044$$

$$f_r = s f_s = 0.044 \times 60 = 2.7 \text{ Hz}$$




$$\bar{I}_{ar}' = \frac{-120\angle 0}{29.3 + j1} = -4.1\angle 0 \quad \text{Amps Rms}$$

$$P_m = 3 \times 4.1^2 \times 28 = 1{,}412 \text{ W}$$

$$\omega_{m\,\text{mech}} = 1720 \frac{\text{REV}}{\text{MIN}}\left(2\pi \frac{\text{RAD}}{\text{REV}}\right)\left(\frac{1\text{ MIN}}{60\text{ sec}}\right) = 180 \text{ RAD/sec}$$

$$T^e = \frac{1{,}412}{180} = 7.8 \text{ Nm}$$

$$P_{AG} = \frac{1{,}412}{1-.044} = 1477 \text{ W}$$

Check $T^e = \dfrac{1477}{2\pi 60\left(\frac{2}{4}\right)} = 7.8 \text{ Nm}$ ✓

$$P_{IN} = P_{AG} + 3\,|\bar{I}_{as}|^2 R_s = 1477 \quad (\text{Because } R_s = 0)$$

$$\bar{I}_{as} = \frac{120\angle 0}{j40} - (-4.1\angle 0) = 4.1 - j3 = 5.1\angle -36^\circ$$

$$pf = \cos(-36^\circ) = 0.81 \text{ lag}$$

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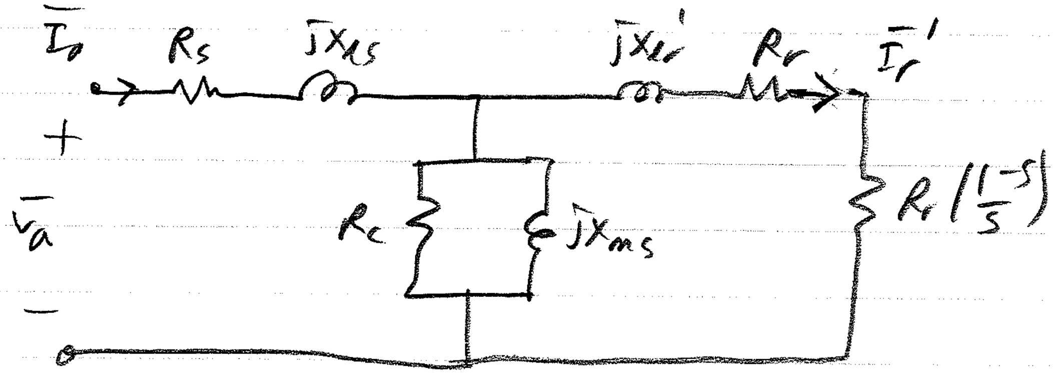
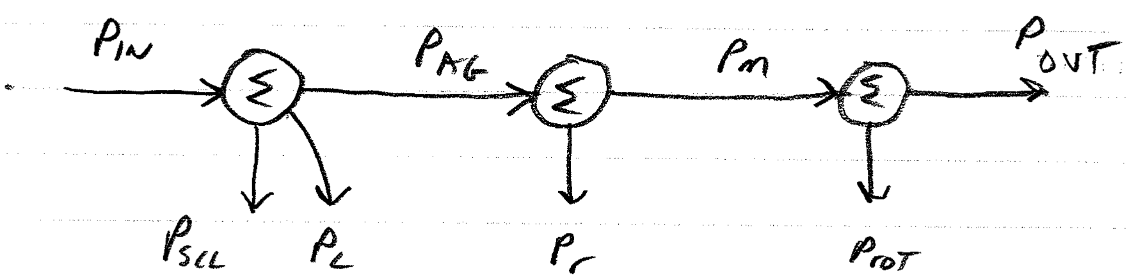



## Efficiency

$$\eta \triangleq \frac{P_{OUT}}{P_{IN}} \times 100$$

$P_{IN} = P_T$ (in book) $\qquad P_{OUT} = P_{shaft}$ (in book)



$$T^e = \frac{P_m}{\omega_{m_{mech}}} = \frac{3|\bar{I}_r'|^2 R_r' \left(\frac{1-s}{s}\right)}{\omega_{m_{mech}}}$$

But recall $\omega_{m_{mech}} = (1-s)\omega_s \left(\frac{2}{p}\right)$

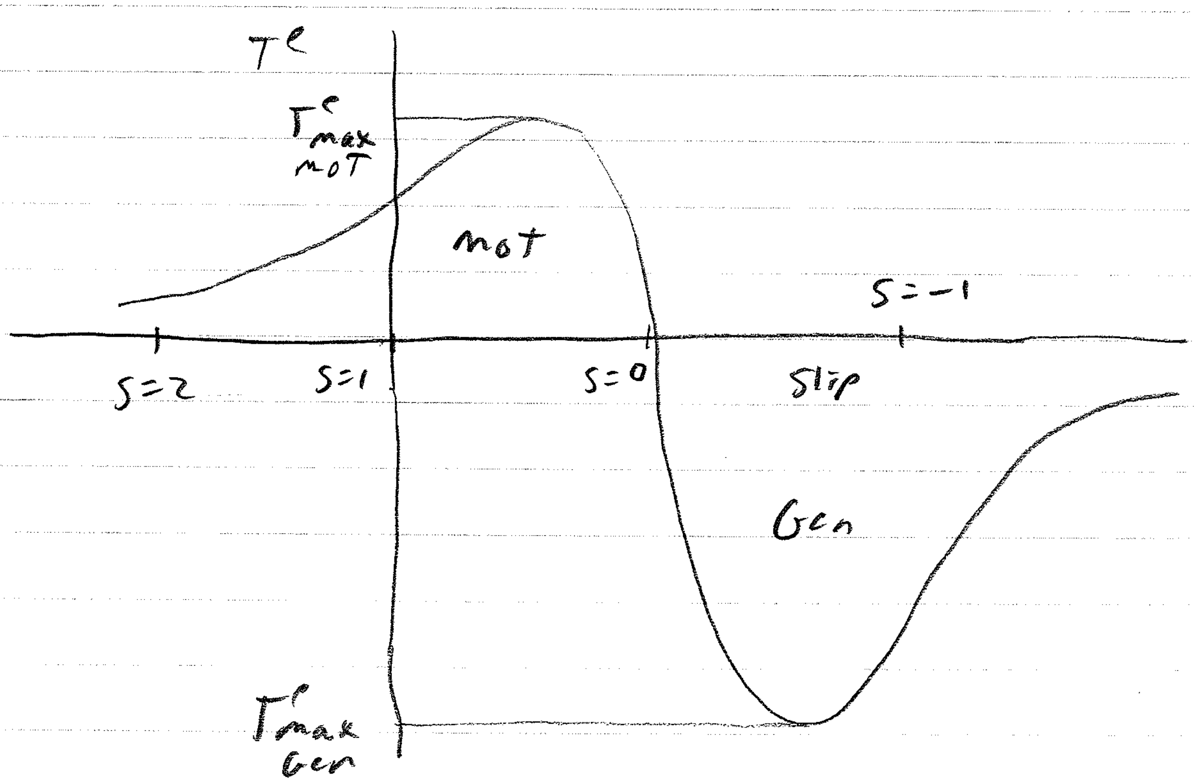
and $P_{AG} = 3|\bar{I}_r'|^2 \frac{R_r'}{s} = P_m/(1-s)$

so:

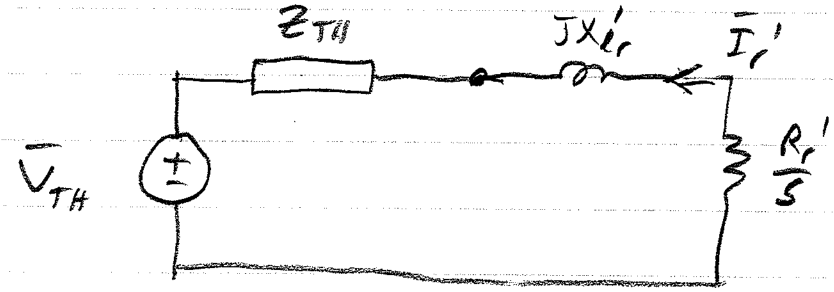
$$T^e = \frac{P_{AG}}{\omega_s\left(\frac{2}{p}\right)} = \frac{3|\bar{I}_r'|^2 \frac{R_r'}{s}}{\omega_s\left(\frac{2}{p}\right)}$$

(Note: $T^e$ is max when $P_{AG}$ is max)

For fixed $|\bar{V}_a|$ and fixed $\omega_s$,



Thevenin equivalent circuit




$$P_{AG} = 3\,|\bar{I}_r'|^2 \frac{R_r'}{s}$$

$$|\bar{I}_r'| = \left| \frac{\bar{V}_{TH}}{\bar{Z}_{TH} + \frac{R_r'}{s} + jX_{lr}'} \right|$$

$$P_{AG} = \frac{3\,|\bar{V}_{TH}|^2 \frac{R_r'}{s}}{\left(R_{TH} + \frac{R_r'}{s}\right)^2 + \left(X_{TH} + X_{lr}'\right)^2}$$

$$T^e = \frac{3\,|\bar{V}_{TH}|^2 \frac{R_r'}{s}}{\omega_s \frac{2}{p}\left[\left(R_{TH} + \frac{R_r'}{s}\right)^2 + \left(X_{TH} + X_{lr}'\right)^2\right]}$$

$\downarrow$

note: for $S<0$ (generator, this term is smaller than for motor.

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$$T^e = \frac{3|\bar{V}_{TH}|^2 R_r' s}{\omega_s \frac{2}{p}\left[(R_{TH} s + R_r')^2 + s^2 (X_{TH} + X_{lr}')^2\right]}$$

For small slip (so $R_{TH} s \ll R_r'$ and $s^2 (X_{TH} + X_{lr}')^2 \ll R_r'^2$

$$T^e \approx \frac{3|\bar{V}_{TH}|^2}{\omega_s \frac{2}{p} R_r'} s$$ (note: book has $\omega_s$, not $\frac{2}{p}\omega_s$)

"Linear part"

But, this does not capture the maxima

Look back at the thevenin equivalent

Power to $P_{AG}$ is max when

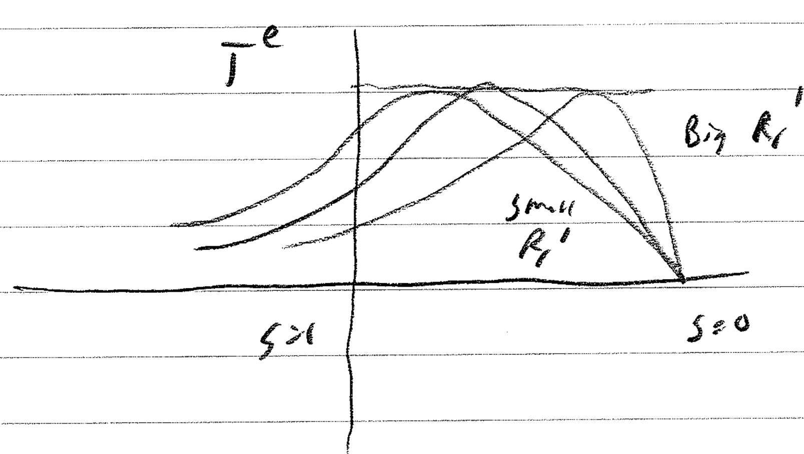
$$\frac{R_r'}{s} = \pm\sqrt{R_{TH}^2 + (X_{TH} + X_{lr}')^2}$$

Slip for maximum Torque is

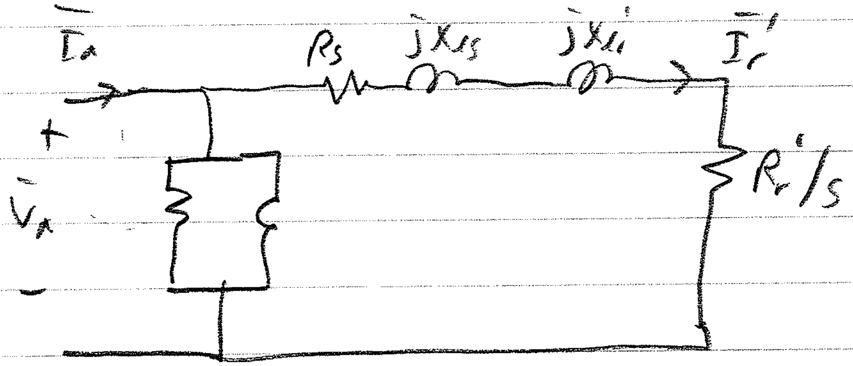
$$s_{\substack{max\\T}} = \pm \frac{R_r'}{\sqrt{R_{TH}^2 + (X_{TH} + X_{lr}')^2}}$$

Substitute into $T^e$ and find that

$T^e_{max}$ is independent of $R_r'$



Using the approximate circuit




$$T^e = \frac{3|\bar{I}'_r|^2 \frac{R'_r}{s}}{\omega_s \frac{2}{p}} \qquad \bar{I}'_r = \frac{\bar{V}_a}{(R_s + \frac{R'_r}{s}) + j(X_{ls} + X'_{lr})}$$

$$T^e = \frac{3\,|\bar{V}_a|^2 R'_r/s}{\omega_s \frac{2}{p}\left[(R_s + \frac{R'_r}{s})^2 + (X_{ls} + X'_{lr})^2\right]}$$

Linear part $\quad T^e \approx \dfrac{3|\bar{V}_a|^2}{\omega_s \frac{2}{p} R'_r} s$

Nonlinear part

Max torque when $\dfrac{R'_r}{s} = \pm\sqrt{R_s^2 + (X_{ls} + X'_{lr})^2}$

$$s_{\max T} = \pm \frac{R'_r}{\sqrt{R_s^2 + (X_{ls} + X'_{lr})^2}}$$

For $R_s = 0$

$$S_{\substack{max \\ T}} = \pm \frac{R_r'}{(X_{ls} + X_{lr}')}$$

$$T^e_{max} = \pm \frac{3|\bar{V}_a|^2 (X_{ls} + X_{lr}')}{\omega_s \frac{2}{P} \left[ 2(X_{ls} + X_{lr}')^2 \right]} = \pm \frac{3 \, |\bar{V}_a|^2}{2 \, \omega_s \frac{2}{P} (X_{ls} + X_{lr}')}$$

Independent of $R_r'$

note: also have $\left| T^e_{\substack{max \\ motor}} \right| = \left| T^e_{\substack{max \\ Gen}} \right|$

Examples

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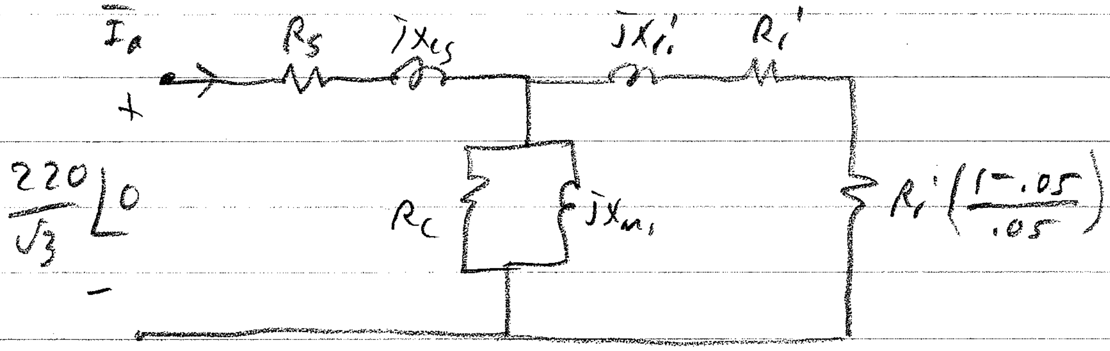
Text Problem 7.4




$pf = .88$ lag

$\bar{I}_a = 77 \angle$ $\qquad$ $P_{SCL} = 1033$ $\quad$ $P_c = 485$

(a) $P_{IN} = 3 \times \frac{220}{\sqrt{3}} \times 77 \times 0.88 = 25{,}821$ W

$P_{AG} = P_{IN} - P_{SCL} - P_c = 24{,}303$ W

(b) $T^e = \frac{P_{AG}}{\omega_s \frac{2}{P}} = \frac{24{,}303}{2\pi 60 \times \frac{2}{4}} = 129$ Nm

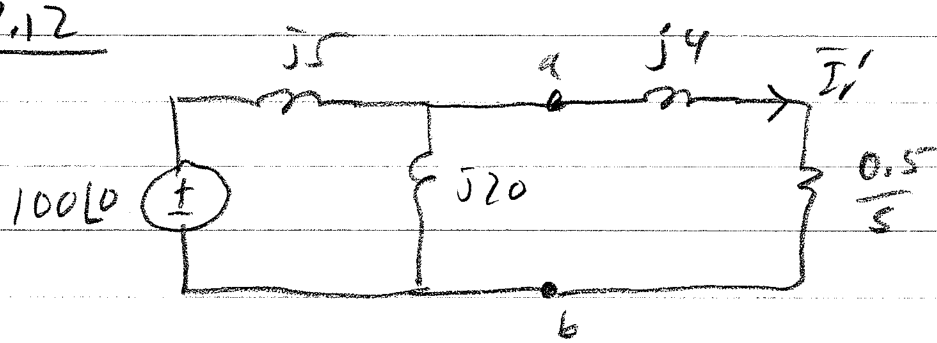
(c) $P_m = (1-s)P_{AG} = .95 \times 24{,}303 = 23{,}088$ W

(d) $P_{OUT} = P_{Shaft} = P_m - P_{rot} = 23{,}088 - 540 = 22{,}548$ W

$HP = \frac{P_{OUT}}{746} = \frac{22{,}548\text{ W}}{746\text{ W}} = 30.2$ HP

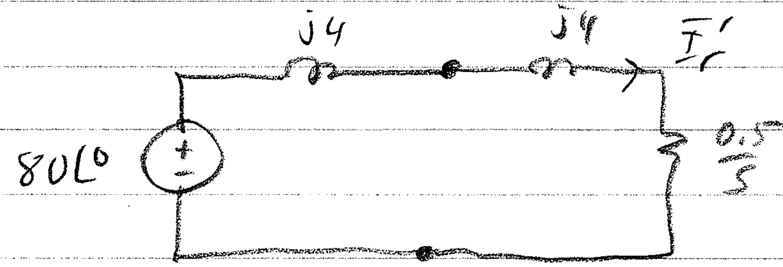
$\eta = \frac{P_{OUT}}{P_{IN}} \times 100 = \frac{22{,}548}{25{,}821} \times 100 = 87\ \%$

7.12



(a) $\bar{V}_{TH_{ab}} = j20\left(\frac{100\angle 0}{j25}\right) = 80\angle 0$

$$\bar{Z}_{TH} = \frac{j5 \times j20}{j25} = j4$$



$$\bar{I}_r' = \frac{80\angle 0}{\frac{0.5}{s} + j8} = \frac{80(s)}{0.5 + j8(s)}$$

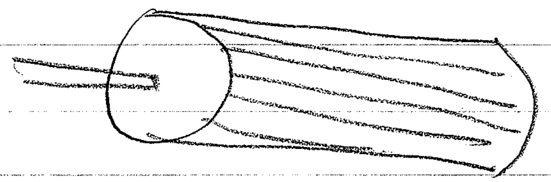
$$T^e = \frac{3 \times |\bar{I}_r'|^2 \frac{0.5}{s}}{2\pi 60 \times \frac{2}{4}} = \frac{3 \times 6400(s^2)\left(\frac{0.5}{s}\right)}{2\pi 30 \times (0.25 + 64s^2)}$$

$$= \frac{51(s)}{.25 + 64(s^2)}$$

(b) $s_{max_T} = \frac{0.5}{8} = .0625$ $\quad T^e_{max} = 6.4$ Nm

$$T^e_{start} = T^e\Big|_{s=1} = 0.8 \text{ Nm}$$

Squirrel Cage rotor



Shorted bars

Equivalent to wound rotor with shorted windings