- Homeworks are due Tuesdays at 5 p.m. Late homework will not be accepted.
- Unstapled homeworks will not be accepted.
- Write your name, netID, and section on each homework.
- Homework are to be turned in to homework boxes on 3rd floor of ECEB. Please put your homework into the appropriate box for the section you are registered in:
  - Section X - 12:00 PM (Xu Chen): Box 35
  - Section E - 1:00 PM (Lynford Goddard): Box 36
  - Section F - 2:00 PM (Yang Zhao): Box 37
- Each student must submit individual solutions for each homework. You may discuss homework problems with other students registered in the course, but you may not copy their solutions. If you use any source outside of class materials that we've provided, you must cite every source that you used.
- Use of homework solutions from past semesters is not allowed and is considered cheating. Copying homework solutions from another student is considered cheating.
- Penalties for cheating on homework: 50% reduction in homework average on first offense, 100% reduction in homework average and report to college on second offense.

Reading Assignment: Kudeki: Lectures 9-12
Recommended Reading: Staelin: 2.5-2.6, 3.1, 3.3-3.5, 4.4-4.5

1. Consider two infinite, plane parallel, perfectly conducting plates at $z = 0$ and $z = z_0 > 0$, which hold equal and opposite surface charge densities and are kept at potentials $V = 0$ and $V = V_p > 0$, respectively. The region between the plates is filled with two slabs of perfect dielectric materials having permittivities ϵ_1 for $0 < z < d$ (region 1) and ϵ_2 for $d < z < z_0$ (region 2).

    a) Find the general solution for the electric potentials, $V = V(z)$, (in terms of V_p, d, z_0, ϵ_1, and ϵ_2) in the two regions by solving Laplace's equation piecewise and enforcing the continuity of $V(z)$ at $z = d$. **Hint:** you will also need to use the fact that there is no surface charge accumulation at a boundary between two perfect dielectrics.

    b) Given that $z_0 = 4d = 2\,\text{m}$, $\epsilon_1 = 3\epsilon_0$, $\epsilon_2 = \epsilon_0$, and $\mathbf{E}(0 < z < d) = -5\hat{z}\,\frac{\text{V}}{\text{m}}$, what is the electrostatic potential V_p on the conductor plate at $z = z_0$?

    c) Given the parameters in part (b) above, what is the surface charge density ρ_s on the plate at $z = z_0$?

2. Consider two conducting plates positioned on $z = 0$ and $z = 5$ m surfaces. The plates are grounded and both have zero potential. In between the plates, on $z = 2\,\text{m}$ surface, there is a uniform and static surface charge of $6\epsilon_o\,\frac{\text{C}}{\text{m}^2}$. The permittivity of the region $z < 2$ m is ϵ_o, whereas it is $2\epsilon_o$ in the region $z > 2$ m. Determine the surface charge densities at $z = 0$ and $z = 5$ m.

    Hint: Let V_o denote the electrostatic potential at $z = 2$. Express the electric and displacement fields above and below $z = 2$ in terms of V_o and use boundary condition equations on $z = 0$, 2, and 5 m surfaces to relate V_o to pertinent surface charge densities.

3. The region between two infinite, plane parallel, perfectly conducting plates at $z = 0$ and $z = z_0$ m is filled with two slabs of perfect dielectric materials having constant electric permittivities ϵ_1 for $0 < z < d$ m (region 1) and ϵ_2 for $d < z < z_0$ (region 2), where $d = 4$ m. The bottom plate held at constant potential V_0, while the top plate is grounded. The electrostatic field is only between the plates, and is known to be

$$\mathbf{E}(z) = \begin{cases} \frac{4\epsilon_2}{\epsilon_1+8\epsilon_2}\hat{z}\,\frac{\text{V}}{\text{m}}, & 0 < z < d \\ \frac{4\epsilon_1}{\epsilon_1+8\epsilon_2}\hat{z}\,\frac{\text{V}}{\text{m}}, & d < z < z_0 \end{cases}$$

    a) Verify that the above field satisfies Maxwell's boundary condition regarding $\mathbf{D}$ at the boundary between the two dielectric slabs.

    b) Write the expression for the electrostatic potential $V(z)$ throughout both regions (i.e., for $0 < z < z_0$ m) in terms of ϵ_1, ϵ_2, z_0, V_0, and d.

    c) Determine ϵ_1 if $\epsilon_2 = 2\epsilon_0$ and the surface charge density on the bottom plate (at $z = 0$ m) is $\rho_s = 4\epsilon_0$ C/m^2.

    d) What is the thickness of region 2 in meters if $V_0 = 3$ V? Use the ϵ_1 and ϵ_2 that you calculated in part (c).

    e) Does $V(z)$ determined in part (b) satisfy Laplace's equation in the region $0 < z < z_0$ m? Explain your answer.

    f) Determine the capacitance C of the structure described above if the parallel plates at $z = 0$ and $z = z_0$ m were constrained to have finite areas $A = W^2$ facing one another (where $W \gg z_0$ so that fringing effects can be neglected). In this calculation ignore the fringing fields, and express C as a function of ϵ_1, ϵ_2, d, z_0 and A.

4. Copper is a highly conducting metal with a **conductivity** of $\sigma = 5.8 \times 10^7$ S/m and a free-electron density of $N_e = 8.45 \times 10^{28}$ m^{-3}.

   a) Determine the resistance R of a copper wire of radius $r = 1.1$ mm, running along the x-axis from $x = 0$ to $x = 125$m using
   $$R = \frac{1}{G} = \frac{d}{A\sigma}$$
   from Lecture 10. A denotes the cross-sectional area of the wire.

   b) What would be the magnitude of electric field $\mathbf{E}(x)$ within the wire of part (a) if the wire were conducting a 1.2 A current? You may assume a uniform current distribution across the wire cross section.

   c) With the electric field determined from part (b), what would be the mean velocity $\mathbf{v}_{ave}$ of an electron in the wire? **Hint:** first deduce the current density $\mathbf{J} = \sigma\mathbf{E}$ in the wire, and then use the fact that $\mathbf{J} = N_e q\mathbf{v}$ in the copper wire, with $q = -e = -1.6 \times 10^{-19}$ C denoting the charge of an electron.

   d) How long would it take the electron in part (c) to drift from one end of the wire to the other? **Hint:** you may be surprised by the result of your calculation.

5. Consider a pair of metallic spherical shells of radii a and $b > a$ with their centers coinciding with the origin of the reference coordinate system. The medium between the shells has permittivity $\epsilon = 2\epsilon_o$ and conductivity $\sigma = 2 \times 10^{-6}$ S/m.

   a) Use the integral form of Gauss's law to relate the static field between the shells to charge Q residing on the (outer) surface of the inner shell with radius a and integrating the field from radius a to b obtain the relation $Q = CV$ where V is the potential drop from the inner to outer shell and
   $$C = 4\pi\epsilon\frac{ab}{b-a}$$

   b) Taking the limit of capacitance C as $b \to \infty$, obtain the capacitance (in pF) of a metallic shell of radius $a = 1$ m embedded in an infinite region of permittivity $\epsilon = 2\epsilon_o$ and conductivity $\sigma = 2 \times 10^{-6}$ S/m.

   c) What is the conductance (in μS) of the metallic shell configuration described in (b). **Hint:** Calculate the Resistance (dR) of a differential spherical shell of radius r and thickness dr. The resistance of the medium between the metallic shells is nothing more than the sum of the resistances (dR) of the differential shells as r goes from $r = a$ to $r = b$. Check your result against the $G = \frac{\sigma}{\epsilon}C$ relation from class notes.

   d) Develop an RC equivalent circuit to represent the physical system described above. Use this equivalent circuit to derive a differential equation that governs the temporal variation of the charge on the spherical shell. **Hint:** in this circuit the lossy capacitor (see Lect 10, page 4) conducts no external current (since external circuit is an effective open in this case — there is no path for the return current).

   e) Assume $Q(t) = 1$ C on the shell with radius a at $t = 0$. The radial electric field produced by $Q(t)$ will drive a radial current to discharge the shell. Obtain an expression for $Q(t)$ in $t > 0$ that describes the discarge of the metallic shell by solving the differential equation obtained in (d).

6. The gap between a pair of parallel infinite copper plates extends from $z = 0$ to $z = W > 0$ and is initially occupied by vacuum (ϵ_o, μ_o). The plates carry equal and oppositely signed surface charge densities and as a consequence we have a constant electric field $\mathbf{E} = 3\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}$ in vacuum in the gap region and zero electric field elsewhere.

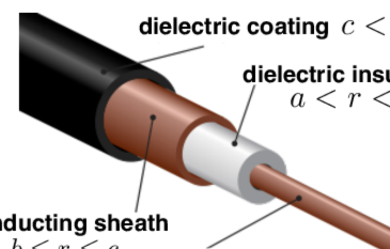
   a) What are the corresponding displacement vector $\mathbf{D}$ and polarization vector $\mathbf{P}$ in the gap region?

   b) What is the surface charge density ρ_s of the copper plate at $z = 0$?

   c) Next, we fill the gap with a non-conducting fluid of permittivity $\epsilon = 81\epsilon_o$ without changing the surface charge densities of the copper plates. What are the new values of $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$ in the gap region?

   d) What would be the new equilibrium values of $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$ in the gap region if some amount of salt were dissolved in the fluid in the gap (see part c) to raise its conductivity to $\sigma = 4\,\frac{\mathrm{S}}{\mathrm{m}}$ (conductivity of sea water)? State the values of $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$ after a steady-state equilibrium is reached and briefly explain your answer.

7. Consider an infinitely long co-axial (cylindrically symmetric) cable, comprised of the following configuration of composite materials in *steady-state equilibrium* (see figure below):
   (i) The center core region, defined by $r \leq a$ (where $r = \sqrt{x^2 + y^2}$ is the radial distance from the origin), is made of a conducting material, having $\epsilon = \epsilon_o$ and $\sigma = 10^6\,\frac{\mathrm{S}}{\mathrm{m}}$, which holds a net charge per unit length of $Q = 2\,\frac{\mathrm{C}}{\mathrm{m}}$.
   (ii) Region $a < r < b$ contains a cylindrical shell made of perfect dielectric material having $\epsilon = 10\epsilon_o$
   (iii) Region $b \leq r \leq c$ is occupied by a shell made of conducting material having the same properties as region $r \leq a$ and holds a net charge per unit length of $-4\,\frac{\mathrm{C}}{\mathrm{m}}$.
   (iv) Region $c < r < d$ contains another perfect dielectric shell having $\epsilon = 2\epsilon_o$.
   (v) Region $r \geq d$ is occupied by free space.



   Determine in all five regions: (a) $\mathbf{D}$, (b) $\mathbf{E}$, and (c) $\mathbf{P}$, *and* determine (d) the surface charge densities, in $\frac{\mathrm{C}}{\mathrm{m}^2}$ units, at each of the four material boundaries at $r = a$, b, c, and d.

   Hint: Make use of Gauss's law in integral form, $\oint_S \mathbf{D} \cdot dS = \int_V \rho dV$, with $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_o\mathbf{E} + \mathbf{P}$, and a crucial fact about steady-state fields within conducting materials.

8. **Bonus Problem:** In an anisotropic material, the permittivity is described by a tensor (matrix) rather than a scalar. Suppose $\epsilon = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \epsilon_0$.

a) For $\mathbf{E} = E_0\hat{y}$, find $\mathbf{D}$ and $\mathbf{P}$ and give a brief physical explanation of what it means for $\mathbf{D}$ and $\mathbf{E}$ to point in the same direction.

b) For $\mathbf{E} = E_0\hat{x}$, find $\mathbf{D}$ and $\mathbf{P}$ and give a brief physical explanation of what it means for $\mathbf{D}$ and $\mathbf{E}$ to not point in the same direction.

c) Find three directions such that $\mathbf{D}$ and $\mathbf{E}$ point in the same direction.