we are now poised to started solving real problems. Most electromagnetic analysis is based on the solution of Maxwell's equations and we have accumulated (almost) enough mathematical tools to begin learning how to use them. BUT FIRST — REVIEW

I) Lorentz equation

\[ \mathbf{F} = q\mathbf{E} + \frac{q}{m} \mathbf{v} \times \mathbf{B} \]

- \( \mathbf{F} \) magnetic field
- \( \mathbf{E} \) electric field
- \( \mathbf{v} \) velocity
- \( q \) charge
- \( m \) mass

Math Tools:

- vector algebra
- vector notation
- concept of fields
Visualization exercise

Find the force on a charged particle if its velocity is:

a) $\vec{v} = 0$ at $(1, 1, 0)$

b) $\vec{v} = -1 \hat{\mathbf{x}}$ at $(0, 1, 0)$

c) $\vec{v} = 1 \hat{\mathbf{z}}$ at $(0, 1, 0)$

d) $\vec{v} = 1 \hat{\mathbf{z}}$ at $(0, 0, 1)$

Given that an external electric and magnetic field exist (but are unknown), design an experiment to determine $\vec{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}$ and $\vec{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$.

Can you find all components?
\[ \vec{F}_2 = \frac{q_1 q_2}{4\pi\varepsilon_0 \left| \vec{r}_2 - \vec{r}_1 \right|^2} \cdot \frac{\vec{r}_2 - \vec{r}_1}{\left| \vec{r}_2 - \vec{r}_1 \right|} \]

written in terms of position vectors:

\[ \vec{r}_1 = x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z} \]
\[ \vec{r}_2 = x_2 \hat{x} + y_2 \hat{y} + z_2 \hat{z} \]

\[ \vec{F}_1 = ? \]

Introduced the electric field \( \vec{E} \)

\[ \vec{F} = q \vec{E} \quad \text{force per unit charge} \]
Find $\mathbf{E}(0,0,0)$.

Set up an integral that computes $\mathbf{E}$ everywhere for this cube of charge.

What is the direction of the force at A?

Its magnitude.
III. GAUSS’ LAW (INTEGRAL FORM)

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \oint \mathbf{D} \cdot d\mathbf{s} = \Phi \]

*the electrical flux... the charge enclosed*

through any closed surface equals...

**MATH TOOLS:**
- surface integrals
- Divergence theorem
i) Can the be solved using the relation $\oint \mathbf{E} \cdot d\mathbf{A} = Q$?

ii) If so, what must $\beta_3$ be?

iii) If so, find $\mathbf{E}$ everywhere.

i) Find $\int \mathbf{E} \cdot d\mathbf{A}$ through the hemisphere $S_1$.

ii) Find $\int \mathbf{E} \cdot d\mathbf{A}$ through $S_2$.

iii) Design a symmetric charge distribution.
Find $\bar{E}$ everywhere.
METHOD II: Gauss’ Law (tediously)

By symmetry \( \mathbf{E} = E_x(x) \mathbf{\hat{x}} \)

Choose Surface of integration

Choose a cylindrical surface with sides parallel to field.

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \int_{s_{left}} \mathbf{E} \cdot d\mathbf{s} + \int_{s_{right}} \mathbf{E} \cdot d\mathbf{s} \]

\[ = \int_{s_{left}} \varepsilon_0 \mathbf{E} \cdot d\mathbf{s} = \int_{s_{right}} \varepsilon_0 \mathbf{E} \cdot d\mathbf{s} \]

\[ d\mathbf{s} = dydz \mathbf{\hat{x}} \]

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \int_{s_{left}} \varepsilon_0 E_x \, dx \, dy \]

Find \( \mathbf{E} \) everywhere:

Method I - the graphical approach.

\[ E_x = E_{x}(x) \mathbf{\hat{x}} \]

i) Argue that by symmetry \( E = E_{x}(x) \mathbf{\hat{x}} \) for \( x > 4 \)

ii) Argue that \( E = 0 \) for \( x < -4 \), WHY?

iii) Find \( \mathbf{E} \) at \( x = 3 \)

to the right of all slabs except the one between \( x = 3 \) and \( x = 4 \).

\[ \mathbf{E} = \mathbf{E}_{\text{right}} + \mathbf{E}_{\text{left}} \]

\[ = \left[ \frac{-2\mathbf{E}_0}{\varepsilon_0} \right] \mathbf{\hat{x}} \]

\[ = -\frac{2\mathbf{E}_0}{\varepsilon_0} \mathbf{\hat{x}} \]

iv) Find \( \mathbf{E} \) at \( x = 2 \).

\[ \mathbf{E} = \left[ \frac{-3\mathbf{E}_0}{\varepsilon_0} \right] \mathbf{\hat{x}} \]

\[ = -\frac{3\mathbf{E}_0}{\varepsilon_0} \mathbf{\hat{x}} \]
METHOD II: (cont.)

\[ \oint \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = - \int_{-4}^{4} \varepsilon_0 \mathbf{E}_1(x) dx + \int_{-3}^{-4} \varepsilon_0 \mathbf{E}_2(x) dx \]

Now we must argue that \( E_{x1} = 0 \)

So that \( \oint \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_{-3}^{-4} \varepsilon_0 \mathbf{E}_2(x) dx \)

\( E_{x2} \) is a constant value on the face so the integral evaluates to:

\[ \varepsilon_0 E_{x2}(x) \cdot \pi a^2 \]

Now find the amount of charge enclosed by the surface.

EQUATE: \( \varepsilon_0 E_{x2}(x) \pi a^2 = \pi a^2 (4+x)(-9_0) \)

\[ E_{x2}(x) = \frac{\int_{-4}^{4} (x+4) dx}{\varepsilon_0} \]

at \( x = -4 \) \( E_{x2} = 0 \)

at \( x = -3 \) \( E_{x2} = -9_0 \)
IV. Gauss' Law (differential form)

\[ \nabla \cdot \vec{D} = \rho \]

- electric flux density, \( \text{C/m}^2 \)

- change

\[ \nabla \cdot \vec{D} = 0 \]

**Math tools:**
- divergence theorem
- operator notation
- concept of divergence
Suppose \( \mathbf{E} = \frac{x}{2^2} \hat{x} + \frac{y}{2^2} \hat{y} \)

determine the charge density \( \rho \) that generated

\[ \mathbf{E} \]

Suppose \( \rho = \rho_0 \frac{x}{W} \)

find \( \mathbf{D} \) using differential form.
Suppose \( E = \frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y} \)

Determine \( \nabla \cdot E = \nabla \cdot (\frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y}) = \frac{2}{2} \frac{x^2}{2} \hat{x} + \frac{2}{2} \frac{y^2}{2} \hat{y} = \frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y} = \frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y} \)

By Gauss' Law
\[ \nabla \cdot E = \nabla \cdot (\frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y}) = \frac{2}{2} \frac{x^2}{2} \hat{x} + \frac{2}{2} \frac{y^2}{2} \hat{y} = \frac{x^2}{2} \hat{x} + \frac{y^2}{2} \hat{y} \]

\[ \nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{2} & \frac{y}{2} & \frac{2}{2} \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{2} & \frac{y}{2} & 0 \end{vmatrix} = \frac{2y}{2} \hat{x} + \frac{2x}{2} \hat{y} \]

Suppose \( g = g_0 - \frac{g_0 x}{W} \) in \( 0 < x < W \)

Find \( D \) using the differential form
\[ \nabla \cdot D = g \]

In region I and III:
\[ \frac{dD_x}{dx} = 0 \Rightarrow D_x = \text{const.} \]

In region II:
\[ \frac{dD_x}{dx} = g_0 - \frac{g_0 x}{W} \]
\[ D_x = g_0 x - \frac{g_0 x^2}{2W} + \text{const.} \]
\[ \nabla \cdot \mathbf{D} = \rho \quad \iff \quad \oint \mathbf{D} \cdot d\mathbf{s} = Q \]
\[ \nabla \times \mathbf{E} = 0 \quad \iff \quad \oint \mathbf{E} \cdot d\mathbf{l} = 0 \]

**Math Tools:**
- Faraday's Law (integral form)
  - Line integral
  - STOKES'S theorem
- Faraday's Law (Differential form)
  - Curl
  - Conservative field
Given a vector field \( \vec{F} = (y+2) \hat{x} + (x+2) \hat{y} + (x+y) \hat{z} \)

\( \text{i) Is it conservative?} \)

\( \text{ii) Can a potential function be found such that,} \)

\[ \vec{F} = \nabla \phi \]

Repeat for \( \vec{F} = (x+2z) \hat{x} + (x+z) \hat{y} + (x+y) \hat{z} \)
Solutions:

Given the vector field \( \mathbf{F} = (x+z) \mathbf{\hat{x}} + (x+z) \mathbf{\hat{y}} + (x+y) \mathbf{\hat{z}} \)

i) Is \( \mathbf{F} \) a conservative field?

**METHOD 1:** take the curl

\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{\hat{x}} & \mathbf{\hat{y}} & \mathbf{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+y & x+y \end{vmatrix}
\]

\[
= \begin{bmatrix} 2(y+z) & 2(y+z) & 0 \\ 0 & 0 & 0 \\ y+z & x+y & x+y \end{bmatrix}
\]

\[
\Rightarrow \text{curl} \mathbf{F} = \begin{bmatrix} 2(y+z) \mathbf{\hat{x}} + 2(y+z) \mathbf{\hat{y}} + 0 \mathbf{\hat{z}} \\ 0 \mathbf{\hat{x}} + 0 \mathbf{\hat{y}} + 0 \mathbf{\hat{z}} \\ y+z \mathbf{\hat{x}} + x+y \mathbf{\hat{y}} + x+y \mathbf{\hat{z}} \end{bmatrix}
\]

\[
\Rightarrow \text{curl} \mathbf{F} \neq 0
\]

\[
\Rightarrow \mathbf{F} \text{ does not describe a conservative field}
\]

**METHOD 2:** find the line integral around a closed path.
drawback - to REALLY REALLY prove that $\mathbf{F}$ is conservative must repeat infinitely many times

METHOD 2: (cont. . .)

Pick a simple path (contour C) and solve $\int_C \mathbf{E} \cdot d\mathbf{l} = 0$

Leg 1: $\mathbf{E} = (y+z) \hat{x} + (x+z) \hat{y} + (x+y) \hat{z}$

$$d\mathbf{l} = dx \hat{x}$$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^1 (y+z) \, dx$$

cancel

did not need to know location of path to find solution.

Leg 2: $d\mathbf{l} = dy \hat{y}$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^1 (x+z) \, dy$$

cancel

Leg 3: $\int \mathbf{E} \cdot d\mathbf{l} = \int_0^0 (y+z) \, dx$

Leg 4: $\int \mathbf{E} \cdot d\mathbf{l} = \int_0^0 (x+z) \, dy$
Method 2: (cont. . . )

Drawback - to really, really prove that \( \mathbf{\nabla} \cdot \mathbf{E} = \mathbf{0} \) conservative must repeat infinitely many times

Pick a simple path (contour C) and solve

\[
\oint_C \mathbf{E} \cdot d\mathbf{l} = 0
\]

Leg 1:

\[
\mathbf{E} = (y+z) \hat{x} + (x+z) \hat{y} + (x+y) \hat{z}
\]

\[
dl = dx \hat{x}
\]

\[
\oint \mathbf{E} \cdot dl = \int_0^1 (y+z) \, dx
\]

Leg 2:

\[
dl = dy \hat{y}
\]

\[
\oint \mathbf{E} \cdot dl = \int_0^1 (x+z) \, dy
\]

Leg 3:

\[
\oint \mathbf{E} \cdot dl = \int_0^1 (y+z) \, dx
\]

Leg 4:

\[
\oint \mathbf{E} \cdot dl = \int_0^1 (x+z) \, dy
\]

did not need to know location of path to find solution.
METHOD 3:

In general \( \int \mathbf{E} \cdot d\mathbf{l} = \int E_x \, dx + E_y \, dy + E_z \, dz \)

for our problem,

\[
= \int (y+z) \, dx + (x+z) \, dy + (x+y) \, dz
\]

Can we find an exact differential?

\[
= \int d\phi(x,y,z)
\]

Finding this special function leads into the next part.
II Voltage -

Because one of Maxwell's Eq. for static electric fields constrains the curl of \( \overline{E} \) to be zero \( \Rightarrow \nabla \times \overline{E} = 0 \), we can use this property to define a scalar function \( V \) to replace \( \overline{E} \):

\[
\overline{E} = -\nabla V
\]

MATH Tools:
- gradient
- potential functions
- exact derivative
If
\[ V(x, y, z) = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \]

find \( \overline{E} \) Sketch \( V(x, y, z) \) in the \( x-y \) plane and \( y-z \) plane.

Draw \( \overline{E} \) superposed on the graph of \( V \).

If \( \overline{E} = E \times \mathbf{k} \) find \( V \)

**Method 1:**

Know relationship between \( V \) and \( \overline{E} \)

\[ \overline{E} = -\nabla V \]

Write out components

\[ E_x = E_0 x \]
\[ \nabla V = -\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial V}{\partial z} \]

Equate components:

\[ E_x = -\frac{\partial V}{\partial x} \Rightarrow V = -E_0 x + f(y, z) \]
\[ E_y = -\frac{\partial V}{\partial y} \Rightarrow V = C_1 + f(x, z) \]
\[ E_z = -\frac{\partial V}{\partial z} \Rightarrow V = C_2 + f(x, y) \]

need to know something else about solution to determine final form.

**Method 2:**

\[ \oint_S \overline{E} \cdot d\overline{l} = \nabla_0 (V(0) - V(l)) \]
\[ \int_A^B E \cdot d\mathbf{l} = \int_{x_0}^{x_b} E_0 \cdot dx = -E_0 \int_{x_0}^{x_b} \mathbf{E} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \cdot \mathbf{X} \\]

choose \( x_b \) to be at the origin
where we know \( V = 0 \) and
\( X_a \) to be any arbitrary \( x \)
\[ V(x) = E_0 \cdot x \]

have a general form for \( V \)
And as a lead-in for the next section—
look what we can do with the differential form of the equations—