University of Illinois Summer 2018

ECE 313: Final Exam

Saturday, August 4, 2018, 8-10 a.m.
ECEB 1013

Name: (in BLOCK CAPITALS) Solutions

NetID: ______________________

Signature: ______________________

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of seven problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading	
1. 15 points	
2. 15 points	
3. 10 points	
4. 18 points	
5. 14 points	
6. 18 points	
7. 10 points	
Total (100 points)	

1. **[15 points]** A bag contains 4 orange and 6 blue t-shirts. Two t-shirts are chosen from the bag at random and moved into a second bag, which already contains one orange t-shirt. Then a t-shirt is chosen from the second bag at random.

(a) Let X denote the number of blue t-shirts moved from the first bag to the second bag. Find the pmf of X.

$X \in \{0,1,2\}$

$\boxed{\begin{matrix} oobb \\ oobb \\ bb \end{matrix}} \xrightarrow{2} \boxed{o}$

$p_X(0) = P\{X=0\} = \binom{4}{2}/\binom{10}{2} = \frac{6}{45} = \frac{2}{15}$

$p_X(1) = P\{X=1\} = \binom{4}{1}\binom{6}{1}/\binom{10}{2} = \frac{24}{45} = \frac{8}{15}$

$p_X(2) = P\{X=2\} = \binom{6}{2}/\binom{10}{2} = \frac{15}{45} = \frac{5}{15}$

(b) What is the probability the t-shirt chosen from the second bag is blue?

$P\{T=b\} = \sum_{k=0}^{2} P\{T=b \mid X=k\}\, P\{X=k\}$

$= 0\, p_X(0) + \frac{1}{3} p_X(1) + \frac{2}{3} p_X(2)$

$= 0\left(\frac{2}{15}\right) + \frac{1}{3}\left(\frac{8}{15}\right) + \frac{2}{3}\left(\frac{5}{15}\right)$

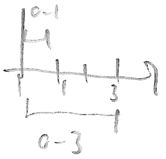
$= \boxed{\frac{2}{5}}$

(c) What is the conditional probability that $X = 2$ given the t-shirt drawn from the second bag is blue?

$P\{X=2 \mid T=b\} = \frac{P\{T=b \mid X=2\}\, P\{X=2\}}{P\{T=b\}} = \frac{\frac{2}{3}\left(\frac{5}{15}\right)}{\frac{2}{5}} = \boxed{\frac{5}{9}}$

2. [**15 points**] Buses arrive at a station starting at time zero according to a Poisson process $(N_t : t \geq 0)$ with rate λ, where N_t denotes the number of buses arriving up to time t.

(a) Given exactly one bus arrives during the interval $[0, 3]$, find the probability it arrives before time $t = 1$. Show your work or explain your reasoning.



[0,1) is a subset of [0,3] and it is $\frac{1}{3}$ of its length $\Rightarrow$ $\boxed{P = \frac{1}{3}}$

Note: You could also use conditional probability.

$$P\{N_1 = 1 \mid N_3 = 1\} = \frac{P\{N_1 = 1, N_3 = 1\}}{P\{N_3 = 1\}} = \frac{P\{N_1 = 1, N_3 - N_1 = 0\}}{P\{N_3 = 1\}}$$

$$= P\{N_1 = 1\} P\{N_3 - N_1 = 0\} / P\{N_3 = 1\}$$

(b) What is the probability that there is a bus arriving exactly at time $t = 1$?

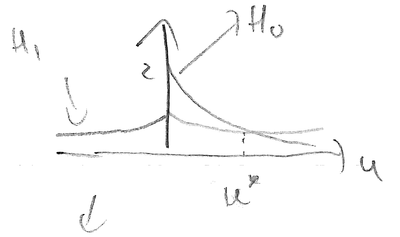
The length of the interval is zero, so $\boxed{P = 0}$.

(c) Suppose Alice arrives at the station at time T, which is uniformly distributed on $[0, 1]$ and independent of the bus arrival process. She decides to wait one unit of time. If no bus arrives while she waits, she walks home. What is the probability Alice ends up walking home?

We can restart the process at T and the first arrival is exponential $\Rightarrow$ $\boxed{P\{U_1 > 1\} = e^{-\lambda}}$

3. [**10 points**] Consider a binary hypothesis testing problem with observation X. Under H_0, $X \sim Exp(2)$, while under H_1, X has pdf $f_X(u) = \frac{1}{2}e^{-|u|}$ for all $u \in (-\infty, \infty)$.

(a) Determine the ML rule.



clearly choose H_1 for $u<0$

For $u>0$

$$\frac{\frac{1}{2}e^{-u}}{2e^{-2u}} \geq 1 \rightarrow H_1$$

$$u \geq \ln 4$$

$$\Rightarrow \boxed{\begin{array}{l} \Gamma_1 = (-\infty, 0] \cup [\ln 4, \infty) \\ \Gamma_0 = (0, \ln 4) \end{array}}$$

(b) Determine $p_{\text{false alarm}}$ for the ML rule.

$$P_{fa} = P\{\text{declare } H_1 \mid H_0\} = \int_{\Gamma_1} f_0(u)\,du = \int_{\ln 4}^{\infty} 2e^{-2u}\,du$$

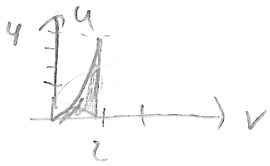
$$= \frac{2e^{-2u}}{-2}\Bigg|_{\ln 4}^{\infty} = \boxed{\frac{1}{16}}$$

4. **[18 points]** Let X and Y be jointly continuous random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} cuv & 0 \le v \le 2, 0 \le u \le v^2 \\ 0 & \text{else,} \end{cases}$$

for a constant c. *You may leave this problem's answers in terms of c, except part (a).*

(a) Determine the value of the constant c.



$$1 = \int_0^2 \int_0^{v^2} cuv\,du\,dv = c\int_0^2 v\frac{u^2}{2}\Big|_0^{v^2} dv$$

$$= \frac{c}{2}\int_0^2 v^5\,dv = \frac{c}{2}\frac{v^6}{6}\Big|_0^2 = \frac{c\cdot 2^6}{2\cdot 2\cdot 3} = \frac{c2^4}{3} = \frac{16c}{3}$$

$$\Rightarrow c = \frac{3}{16}$$

(b) Obtain the marginal pdf $f_X(u)$ for all u.

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv = \int_{\sqrt{u}}^{2} cuv\,dv = cu\frac{v^2}{2}\Big|_{\sqrt{u}}^{2}$$

$$\boxed{= \begin{cases} \frac{c}{2}u(4-u) & u \in (0,4) \\ 0 & \text{else} \end{cases}}$$

(c) Obtain the conditional pdf $f_{Y|X}(v|u)$ for all u and v (indicate where it is undefined).

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \boxed{\begin{cases} \text{undefined} & \text{if } u \notin (0,4) \\ \frac{2v}{4-u} & u \in (0,4), v \in (\sqrt{u}, 2) \\ 0 & \text{else} \end{cases}}$$

(d) Are X and Y independent? Explain why or why not.

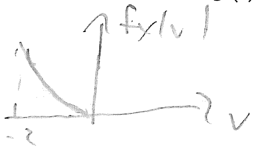
$\boxed{\text{No}}$, support is not a product set.

(e) Obtain $E\left[\frac{1}{XY}\right]$. $= \int_0^2 \int_0^{v^2} \frac{1}{uv} cuv \, du \, dv = c \int_0^2 v^2 \, dv = c \frac{v^3}{3} \Big|_0^2$

$= \boxed{c \frac{8}{3}}$

5. [**14 points**] The two parts of this problem are unrelated.

(a) Let $U \sim Uniform(0,1)$, and let X have pdf $f_X(v) = \frac{3}{8}v^2$ for $-2 \le v \le 0$. Obtain a function $g(\cdot)$ such that $g(U)$ has the same pdf as X.

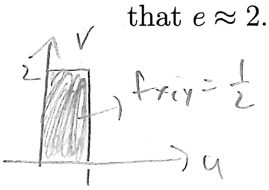


$$F_X(v) = \int_{-\infty}^{v} f_X(s)\,ds = \begin{cases} 0 & v < -2 \\ \int_{-2}^{v} \frac{3}{8} s^2\, ds = \frac{1}{8}(v^3+8) & -2 < v < 0 \\ 1 & v > 0 \end{cases}$$

$$c = \frac{1}{8}(v^3+8) \Rightarrow (8c-8)^{\frac{1}{3}} = v = F_X^{-1}(c)$$

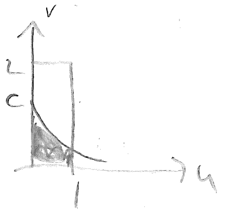
$$\Rightarrow g(U) = F_X^{-1}(U) = (8U-8)^{\frac{1}{3}} = \boxed{2(U-1)^{\frac{1}{3}} = g(U)}$$

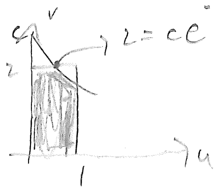
(b) Let X and Y be jointly uniform on the set $S = [0,1] \times [0,2] = \{0 \le u \le 1, 0 \le v \le 2\}$. Let $Z = Ye^X$ and obtain the CDF of Z, $F_Z(c)$, for all c. (Note: recall that $e \approx 2.72$.)



$$F_Z(c) = P\{Z \le c\} = P\{Ye^X \le c\} = P\{Y \le ce^{-X}\}$$

$$= \begin{cases} 0 & c < 0 \\ \int_0^1 \int_0^{ce^{-u}} \frac{1}{2}\, dv\, du = \frac{1}{2}\int_0^1 ce^{-u}\, du = \frac{c}{2}(1-e^{-1}) & 0 \le c \le 2 \\ 1 - \int_{\ln(\frac{c}{2})}^{1} \int_{ce^{-u}}^{2} \frac{1}{2}\, dv\, du = 1 - \frac{1}{2}\int_{\ln(\frac{c}{2})}^{1} (2 - ce^{-u})\, du & \\ \quad = 1 - \frac{1}{2}\left[2(1-\ln(\frac{c}{2})) - c(\frac{2}{c} - e^{-1})\right] & \\ \quad = \ln(\frac{c}{2}) + 1 - \frac{c}{2}e^{-1} & 2 < c \le 2e \\ 1 & c > 2e \end{cases}$$





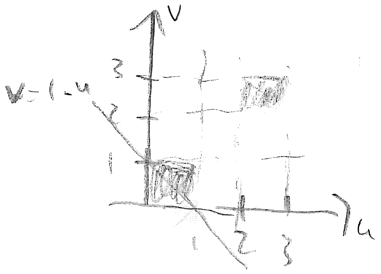
$z = ce^{-u} \Rightarrow u = \ln(\frac{c}{2})$

> 1

$c > 2e^{1}$

6. **[18 points]** Let X and Y be jointly uniform on the set $S = [0,1]\times[0,1]\bigcup[2,3]\times[2,3] = \{\{0 \le u \le 1, 0 \le v \le 1\}\bigcup\{2 \le u \le 3, 2 \le v \le 3\}$.

(a) Obtain $P\{X+Y \ge 1\}$. $= P\{Y \geq 1-X\} = \int_0^1 \int_0^{1-u} \frac{1}{2}\, dv\, du + \int_2^3 \int_2^3 \frac{1}{2}\, dv\, du$

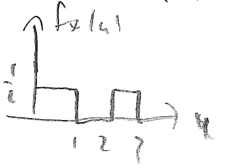


$f_{XY} = \frac{1}{2}$

$= \left[\frac{(1)(1)}{2} + (1)(1)\right]\frac{1}{2}$

$= \boxed{\frac{3}{4}}$

(b) Obtain $Cov(X,Y)$ and $Cov(-2X+3, 4Y-2X)$.



$\mu_X = \frac{3}{2} = \mu_Y$

$E[X^2] = \int_0^1 u^2 \frac{1}{2}\, du + \int_2^3 u^2 \frac{1}{2} = \frac{1}{2}\left[\frac{1}{3} + \frac{27-8}{3}\right] = \frac{20}{6} = \frac{10}{3}$

$\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{10}{3} - \left(\frac{3}{2}\right)^2 = \frac{10}{3} - \frac{9}{4} = \frac{40-27}{12} = \frac{13}{12}$

Same for Y.

$E[XY] = \int_0^1\int_0^1 uv\frac{1}{2} + \int_2^3\int_2^3 uv\frac{1}{2} = \frac{1}{2}\left[\int_0^1 u\frac{1}{2}\, du + \int_2^3 u\left(\frac{9-4}{2}\right) du\right]$

$= \frac{1}{4}\left[\frac{1}{2} + \frac{5}{2}(9-4)\right] = \frac{1}{8}(1+25) = \frac{26}{8} = \frac{13}{4}$

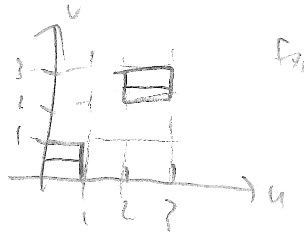
$Cov(X,Y) = E[XY] - \mu_X\mu_Y = \frac{13}{4} - \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \boxed{1}$

$Cov(-2X+3, 4Y-2X) = -2(4)Cov(X,Y) + (-2)^2\sigma_X^2 = -8(1) + 4\left(\frac{13}{12}\right)$

$= \frac{-24+13}{3} = \boxed{\frac{-11}{3}}$

(c) Obtain the best MMSE unconstrained estimator of Y from X, $\hat{Y} = g^*(X)$ and its corresponding $MMSE_{g^*}$.

$g^*(y) = E[Y|X]$



$f_{X,Y} = \frac{1}{2}$

Notice $f_{Y|X} \sim$ Uniform $\Rightarrow E[Y|X=u]$ is halfway up

$$\boxed{g^*(x) = \begin{cases} \frac{1}{2} & x \in (0,1] \\ \frac{5}{2} & x \in (2,3] \end{cases}}$$

$$MMSE_{g^*} = \int Var(Y|X=u) f_X(u)\,du = \int_0^1 \frac{(1-0)^2}{12}\frac{1}{2}\,du + \int_2^3 \frac{(3-2)^2}{12}\frac{1}{2}\,du$$
$$= \boxed{\frac{1}{12}}$$

(d) Obtain the best MMSE linear estimator of Y from X, $\hat{Y} = L^*(X) = \hat{E}[Y|X]$ and its corresponding $MMSE_{L^*}$.

$$\hat{E}[Y|X] = \frac{Cov(Y,X)}{Var(X)}(X - \mu_X) + \mu_Y$$

$$= \frac{1}{\frac{13}{12}}\left(X - \frac{3}{2}\right) + \frac{3}{2} = \boxed{\frac{12}{13}\left(X - \frac{3}{2}\right) + \frac{3}{2}}$$

$$MMSE_{L^*} = \sigma_Y^2 - \frac{Cov^2(X,Y)}{Var(X)} = \frac{13}{12} - \frac{(1)^2}{13/12} = \frac{13}{12} - \frac{12}{13}$$
$$= \frac{169 - 144}{12 \cdot 13} = \boxed{\frac{25}{156}}$$

7. **[10 points]** (2 points per answer)
No partial credit. In order to discourage guessing, 2 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let X and Y be jointly random variables

TRUE	FALSE	
☐	☒	If X and Y are independent then $E\left[Ye^X\right] = E[Y]e^{E[X]}$.

$E[Ye^X] = E[Y]E[e^X]$

TRUE	FALSE	
☒	☐	If $Var(X+Y) = Var(X) + Var(Y)$ then X and Y are uncorrelated.

$= Var(X) + Var(Y) + 2\,Cov(X,Y)$

TRUE	FALSE	
☐	☒	If $f_{Y\|X}(0\|0) = f_Y(0)$ then X and Y are independent.

Has to hold for all y and
x where $f_X(x) \neq 0$

TRUE	FALSE	
☐	☒	If $F_{X,Y}(1,2) = 0.5$ and $F_{X,Y}(1,1) = 0.25$ then $P\{1 < Y \leq 2\} = 0.25$.

$P\{1 < Y \leq 2\} = F_{X,Y}(\infty,2) - F_{X,Y}(\infty,1)$

TRUE	FALSE	
☐	☒	If X and Y are uncorrelated, then the best unconstrained MMSE estimator $g^*(X) = E[Y]$.

Only if independent.