University of Illinois Summer 2018

ECE 313: Exam III

Wednesday, July 25, 2018
10-10.50 a.m.
1013 ECEB

Name: (in BLOCK CAPITALS) Solutions

NetID: ______

Signature: ______

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

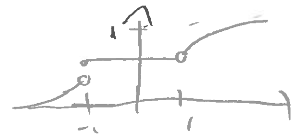
The exam consists of **four** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

Grading	
1. 25 points	
2. 20 points	
3. 25 points	
4. 30 points	
Total (100 points)	

1. **[25 points]** Let X be a random variable with CDF given below, where c_1 is a constant.

$$F_X(u) = \begin{cases} \frac{1}{8}e^{c_1 u+1} & u < -1 \\ \frac{1}{4} & -1 \le u < 1 \\ 1 - \frac{3}{4u} & u \ge 1 \end{cases}$$



NOTE: you can leave your answers in terms of c_1, except for part (a).

(a) Obtain all possible values of c_1.

$0 \le \frac{1}{8}e^{c_1 u+1} \le \frac{1}{4}$ decreasing

as $u \to -\infty$ $\to 0$ $\Rightarrow c_1 > 0$

as $u \to -1^-$ $\Rightarrow \frac{1}{8}e^{-c_1+1} \le \frac{1}{4}$

$-c_1 + 1 \le \ln\left(\frac{8}{4}\right) \Rightarrow c_1 \ge \underbrace{1 - \ln(2)}_{>0}$

$\boxed{c_1 \ge 1 - \ln(2)}$

(b) Obtain $P\{X = 2\}$.

CDF continuous @ $u=2$ $\Rightarrow$ $\boxed{P\{X=2\} = 0}$

Notice that unless $c_1 = 1 - \ln(2)$, then X is not continuous-type.

(c) Obtain $P\{-1 < X < 1\}$. $= F_X(1^-) - F_X(-1) = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

(d) Obtain $P\{X \ge 1\}$. $= 1 - F_X(1^-) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$

2. **[20 points]** The two parts of this problem are unrelated

(a) Let $X \sim N(-1, 4)$. Determine $P\{X^3 < 27\}$ in terms of the Q function.

$$P\{X^3 < 27\} = P\{X < (27)^{\frac{1}{3}}\} = P\{X < 3\}$$

$$= \Phi\left(\frac{3-(-1)}{\sqrt{4}}\right) = \Phi(2) = \boxed{Q(-2)}$$

$$= 1 - Q(2)$$

(b) Let $Y \sim \mathit{Binomial}(1000, 0.25)$. Use the Gaussian approximation with continuity correction to determine $P\{|Y - 250| < 60\}$ in terms of the Q function.

$$P\{|Y-250| < 60\} = P\{-60 < Y-250 < 60\}$$

$$= P\{-59 \leq Y-250 \leq 59\}$$

$$\approx P\{-59.5 < X-250 < 59.5\}$$

$$\downarrow$$

$$N\left(0, 250\left(\tfrac{3}{4}\right)\right) \qquad \overset{125}{\cancel{250}}\left(\frac{3}{\underset{2}{\cancel{4}}}\right) = \frac{375}{2}$$

$$= Q\left(\frac{-59.5}{\sqrt{375/2}}\right) - Q\left(\frac{+59.5}{\sqrt{375/2}}\right)$$

3. **[25 points]** A celebrity couple, Alex and Morgan, Tweets regularly. The time that it takes for their Tweets to be taken down depends on who actually Tweets. If Alex Tweets, the time that it takes for the Tweet to be taken down is uniformly distributed from zero to three hours, while the time that it takes for a Tweet from Morgan to be taken down is exponentially distributed with parameter 2/hour. Assume that each Tweet has probability $\frac{1}{3}$ of being posted by Alex.

A: Uniform$(0,3)$
M: Exp(2) $\quad P_A = \frac{1}{3}$

(a) Suppose a Tweet is posted and it has not been taken down after four hours. What is the conditional probability it will still not be taken down for at least three more hours?

has to be from Morgan

$$P\{T > 4+3 \mid T > 4, M\} = P\{T_M > 3\} = e^{-3(2)} = \boxed{e^{-6}}$$

(b) Suppose another Tweet is posted and it has not been taken down after two hours. What is the conditional probability it will still not be taken down for at least three more hours?

$$P\{T > 2+3 \mid T > 2\} = \frac{P\{T > 2+3, T > 2\}}{P\{T > 2\}} = \frac{P\{T > 5\}}{P\{T > 2\}}$$

$$= \frac{P\{T_A > 5\}P\{A\} + P\{T_M > 5\}P\{M\}}{P\{T_A > 2\}P\{A\} + P\{T_M > 2\}P\{M\}} = \frac{0 + e^{-2(5)}\left(\frac{2}{3}\right)}{\frac{1}{3}\left(\frac{1}{3}\right) + e^{-2(2)}\left(\frac{2}{3}\right)}$$

$$= \boxed{\frac{6e^{-10}}{1+6e^{-4}}}$$

(c) Determine the expected value of the time that it takes for one of their Tweets to be taken down.

$$E[T] = E[T \mid A]P\{A\} + E[T \mid M]P\{M\}$$

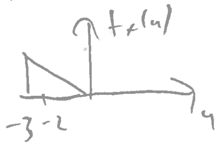
$$= \frac{3}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \boxed{\frac{5}{6}}$$

4. **[30 points]** Let X have pdf $f_X(u) = -\frac{2}{9}u$ for $u \in [-3, 0]$ and zero else.

(a) Determine $P\{X = -2\}$.

X is continuous type $\Rightarrow$ $\boxed{P\{X=-2\}=0}$

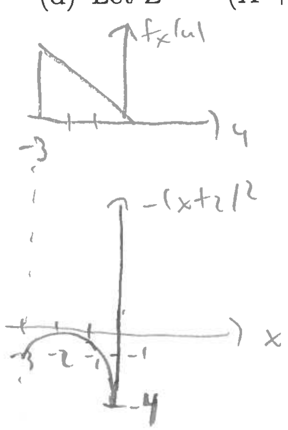
(b) Determine $P\{-2 < X \le 1\}$. $= \int_{-2}^{0} \left(-\frac{2}{9}u\right) du = \frac{2}{2}\left(\frac{4}{9}\right) = \boxed{\frac{4}{9}}$



(c) Determine $E\left[\frac{1}{X}\right]$. $= \int_{-\infty}^{\infty} \frac{1}{u} f_X(u)\, du = \int_{-3}^{0} \frac{1}{u}\left(-\frac{2}{9}u\right) du = -\frac{2}{9}(+3)$

$= -\frac{6}{9} = \boxed{-\frac{2}{3}}$

(d) Let $Z = -(X+2)^2$, determine its pdf, $f_Z(c)$, for all c.



If $c \in (-1,-4]$

$F_Z(c) = P\{Z \le c\} = P\{-(X+2)^2 \le c\}$

$= P\{X \ge +\sqrt{-c} - 2\} = 1 - F_X(\sqrt{-c} - 2)$

$\Rightarrow f_Z(c) = \frac{d}{dt} F_Z(c) = -f_X(\sqrt{-c}-2)\frac{1}{2\sqrt{-c}}(-1)$

$= \frac{1}{2\sqrt{-c}}\left(-\frac{2}{9}(\sqrt{-c}-2)\right) = \boxed{\frac{2-\sqrt{-c}}{9\sqrt{-c}} \quad c \in (-1,-4)}$

$Z \in (-4, 0]$

continuous-type

If $c \in (-1, 0]$

$F_Z(c) = P\{X \le -\sqrt{-c} - 2\} + P\{X \ge \sqrt{-c} - 2\}$

$= F_X(-\sqrt{-c} - 2) + 1 - F_X(\sqrt{-c} - 2)$

$\Rightarrow f_Z(c) = f_X(-\sqrt{-c} - 2)\left[-\frac{1}{2\sqrt{-c}}(-1)\right] + \frac{2-\sqrt{-c}}{9\sqrt{-c}}$

$= \boxed{\frac{4}{9\sqrt{-c}} \quad c \in (-1, 0)}$