University of Illinois Summer 2018

ECE 313: Exam II

Wednesday, July 11, 2018
10 - 10.50 a.m.
1013 ECEB

Name: (in **BLOCK CAPITALS**) Solutions

NetID: ____________________

Signature: ____________________

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of **4** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

Grading	
1. 27 points	
2. 23 points	
3. 20 points	
4. 30 points	
Total (100 points)	

1. **[27 points]** Suppose that you have three biased coins and a fair die. Coin C_1 has $P\{heads\} = 1/3$, while coins C_2 and C_3 have $P\{heads\} = 1/4$. Flip coin C_1 10 times and let X_1 be the total number of heads that show. Flip coin C_2 5 times and let X_2 be the total number of heads that show. Let $Y = X_1 + X_2$.

(a) Determine $E[X_1]$ and $E[Y]$.

$$X_1 \sim \text{Binomial}(10, \tfrac{1}{3}) \rightarrow E[X_1] = 10\left(\tfrac{1}{3}\right) = \boxed{\tfrac{10}{3}}$$

$$X_2 \sim \text{Binomial}(5, \tfrac{1}{4}) \rightarrow E[X_2] = 5\left(\tfrac{1}{4}\right) = \tfrac{5}{4}$$

$$E[Y] = E[X_1 + X_2] = E[X_1] + E[X_2] = \tfrac{10}{3} + \tfrac{5}{4} = \boxed{\tfrac{55}{12}}$$

by linearity of expectation

(b) Obtain $P\{X_1 = 1\}$ and $P\{Y = 1\}$.

$$X_1 \sim \text{Binomial}(10, \tfrac{1}{3}) \rightarrow P_{X_1}(1) = \binom{10}{1}\left(\tfrac{1}{3}\right)^1\left(1-\tfrac{1}{3}\right)^{10-1} = \boxed{\tfrac{10 \cdot 2^9}{3^{10}}}$$

Using total probability

$$P\{Y=1\} = \sum_{k=0}^{10} P\{Y=1 | X_1 = k\} P\{X_1 = k\} = \underbrace{P\{Y=1|X_1=0\}}_{=P\{X_2=1\}} P\{X_1=0\} + \underbrace{P\{Y=1|X_1=1\}}_{=P\{X_2=0\}} P\{X_1=1\}$$

$$= \left[\binom{5}{1}\left(\tfrac{1}{4}\right)^1\left(1-\tfrac{1}{4}\right)^{5-1}\right]\left[\binom{10}{0}\left(\tfrac{1}{3}\right)^0\left(1-\tfrac{1}{3}\right)^{10-0}\right] + \left[\binom{5}{0}\left(\tfrac{1}{4}\right)^0\left(1-\tfrac{1}{4}\right)^{5-0}\right]\left[\binom{10}{1}\left(\tfrac{1}{3}\right)^1\left(1-\tfrac{1}{3}\right)^{10-1}\right]$$

$$= 5 \cdot 3^{-6} \cdot 2^{10-10} + 3^{5-10} \cdot 10 \cdot 2^{9-10} = \boxed{\tfrac{20}{3^6}}$$

(c) Roll the die and let N be the number showing. Then flip coin C_3 N times. Let Z be the number of heads showing. Find $E[Z]$.

Using total probability

$$E[Z] = \sum_{k=1}^{6} E[Z | N=k] \underbrace{P\{N=k\}}_{=\frac{1}{6}} = \frac{1}{6}\sum_{k=1}^{6} E[W_k]$$

$\hookrightarrow \sim \text{Binomial}(k, \tfrac{1}{4})$

$$= \frac{1}{6}\sum_{k=1}^{6} \frac{k}{4} = \frac{1}{24} \cdot \frac{6(7)}{2} = \frac{7}{8}$$

2. **[23 points]** Suppose a fair die is rolled. Let X be the number of times the die is rolled until an even number shows. Let N be that first even number that shows, and let Y be the number of additional times the die is rolled until N shows again.

(a) Determine the pmf of X, and its mean μ_X.

$X \sim \text{Geometric}\left(\frac{1}{2}\right)$

$$\boxed{p_X(k) = \begin{cases} \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k & k \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}}$$

$$\mu_X = \frac{1}{\frac{1}{2}} = \boxed{2}$$

(b) Determine the pmf of Y, and its mean μ_Y.

$Y \sim \text{Geometric}\left(\frac{1}{6}\right)$

$$\boxed{p_Y(k) = \begin{cases} \left(1-\frac{1}{6}\right)^{k-1}\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) & k \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}}$$

$$\mu_Y = \frac{1}{\frac{1}{6}} = \boxed{6}$$

(c) Determine $E[X - Y]$.

$$E[X-Y] = E[X] - E[Y] = 2 - 6 = \boxed{-4}$$

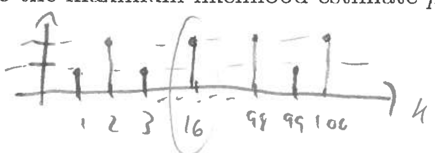
by linearity of expectation

3. **[20 points]** The two parts of this problem are unrelated.

(a) Le X be a random variable with pmf

$$p_X(k) = \begin{cases} \frac{1}{50}p & k \in \{1,3,5,\ldots,100\} \\ \frac{1}{50}(1-p) & k \in \{2,4,6,\ldots,100\}, \end{cases}$$

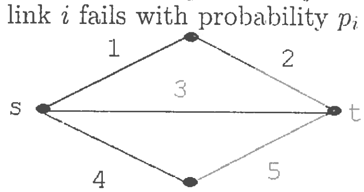
where p is unknown. The experiment is performed once and it is observed that $X = 16$. Determine the maximum likelihood estimate $\hat{p}_{ML}$.



maximize $\frac{1}{50}(1-p) = p_X(16)$ $\rightarrow$ max @ $p=0$

$$\boxed{\hat{p}_{ML} = 0}$$

(b) Determine the probability of network outage in the $s-t$ network below, assuming that link i fails with probability p_i independently of other links.



Route 1 uses 1 & 2

Route 3 uses 3

Route 4 uses 4 & 5

Need all routes to fail

$F_i = \{\text{link } i \text{ fails}\}$

$P\{\text{outage}\} = P\{(F_1 \cup F_2) F_3 (F_4 \cup F_5)\}$

$= P\{F_1 \cup F_2\} P\{F_3\} P\{F_4 \cup F_5\}$ by independence

$$\boxed{= (p_1 + p_2 - p_1 p_2)\, p_3\, (p_4 + p_5 - p_4 p_5)}$$

4. **[30 points]** Consider a binary hypothesis testing problem with the following likelihood matrix:

	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
H_0	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	x	$\frac{5}{12}$
H_1	$\frac{2}{12}$	$\frac{1}{12}$	y	$\frac{3}{12}$	$\frac{4}{12}$

(a) Determine the values of x and y.

Need rows to add up to 1

$\Rightarrow$ $x = 1 - \left(\frac{1}{12} + \frac{2}{12} + \frac{4}{12} + \frac{5}{12}\right) = \boxed{0 = x}$

$y = 1 - \left(\frac{2}{12} + \frac{1}{12} + \frac{3}{12} + \frac{4}{12}\right) = \frac{2}{12} = \boxed{\frac{1}{6} = y}$

(b) Determine the ML rule.

$$\boxed{\begin{array}{l}\Gamma_0 = \{2,3,5\} \\ \Gamma_1 = \{1,4\}\end{array}}$$

by choosing largest entry in each column

(c) Obtain a decision rule such that $p_{miss} = \frac{1}{12}$.

$P_{miss} = P\{\text{declare } H_0 \mid H_1\} =$ sum of unselected entries in H_1 row $= \frac{1}{12}$

$$\Rightarrow \boxed{\begin{array}{l}\Gamma_0 = \{2\} \\ \Gamma_1 = \{1,3,4,5\}\end{array}}$$

(d) If the MAP rule is used, for which value(s) of prior π_1 would H_1 always be declared?

MAP $\quad \Lambda(k) \geq \frac{\pi_0}{\pi_1} = \frac{1-\pi_1}{\pi_1}$

To always declare H_1 this must be true for all k

$\Lambda(k) =$	2	$\frac{1}{2}$	$\frac{2}{4}$	undefined but always choose H_1	$\frac{4}{5}$

smallest, so need

$\frac{1}{2} \geq \frac{1-\pi_1}{\pi_1}$

$\pi_1 \geq 2(1-\pi_1)$

$\pi_1(1+2) \geq 2$

$\boxed{1 \geq \pi_1 \geq \frac{2}{3}}$