ECE 313: Exam I
Wednesday, June 27, 2018
10 - 10.50 a.m.
1013 ECEB

Name: (in BLOCK CAPITALS) ________________________

NetID: __________________________________________

Signature: _______________________________________

Instructions
This exam is printed double-sided, so make sure to look at all problems and both sides of every sheet.

This exam is closed book and closed notes except that one 8.5" × 11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of four problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

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1. [15 points] (3 points per answer) 
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score. For each of the following statements, determine if it is true or false for any three events $A$, $B$ and $C$ in a common probability space.

TRUE  FALSE

- $P(A|B) + P(B) + P(C) \leq 1.$
  - e.g., $P(\emptyset) = 0, \emptyset = P(C)$

- $P(A|B) \cdot P(A|B^c) \leq 1.$
  - Assume $70$

- If $A$ and $B$ are mutually exclusive, then $P(AC) = P(A)P(C)$.
  - Nothing to do with $B$.
  - If you thought it was $A \& C$, it is still false.

- If $E_1, \ldots, E_n$ is a partition of $\Omega$, then $\sum_{i=1}^{n} P(B|E_i) = P(B)$.
  - Law of total probability missing $p(f)$

- If $E_1, \ldots, E_n$ is a partition of $\Omega$, then $\sum_{i=1}^{n} P(E_i|B)P(B) = P(B)$.
  - Conditional probability must still have $p(\emptyset | B) = 1$
2. [25 points] The two parts of this problem are unrelated.

(a) Consider the grid of points below. Suppose that starting at the point labeled A, you can
go one step up or one step to the right on the grid at each move. This continues
until point B is reached. If you can’t step up anymore, then you keep stepping right. If
you can’t step right anymore, then you keep stepping up. How many different paths are
there from A to B?

Notice that we need 9 steps right and
4 steps up, but their order does not matter,
so there are \(\binom{9+4}{9}\) different paths.

\[
\binom{13}{9} = \frac{13!}{9!4!} = \frac{13(12)(11)(10)}{4!} \approx 715
\]

(b) Consider an 8x8 chess board and suppose you have 8 chess pieces. Determine the prob-
ability that none of the eight pieces share the same row, nor the same column.

All placements of the eight pieces are
equally likely.

\[|\pi| = \binom{64}{8}\] because there are 8x8 = 64 spaces
to put eight pieces.

Now, for our specific event, take the first
column and choose one of its 8 rows to
place a piece. Then, take the second row
and choose one of the 7 rows without a piece,
on so forth. So there are 8! choices
where pieces don’t share rows nor columns.

So, \[\frac{8! \binom{64}{8}}{64!} \approx 8! \times 56! \]
3. **[30 points]** Consider and ECE 313 student who is taking a final exam. If the student studied consistently during the semester, which happens with probability 0.3, then he/she has a 90% chance of getting an A in the course. If the student only studied during the last week of the semester, which happens with probability 0.6, then he/she has a 30% chance of getting an A in the course. If the student only studied the day before the exam, which happens with probability 0.1, then he/she has a 1% chance of getting an A in the course. Let $A$ denote the event that the student got an A in the course.

(a) Determine $P(A)$.

Let $S$ = studied whole semester,
    $W$ = studied last week only,
    $D$ = studied last day only.

$P(A) = P(A|S)P(S) + P(A|W)P(W) + P(A|D)P(D)$

$= 0.9(0.3) + 0.3(0.6) + 0.01(0.1)$

$= 0.27 + 0.18 + 0.001 = 0.451 \approx \frac{451}{1000}$

(b) Determine $P\{\text{student studied only the day before the exam}|A\}$.

$P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{0.01(0.1)}{0.451} = \frac{0.001}{0.451}$

$= \frac{1}{451}$

(c) Determine $P\{\text{student studied more than just the day before the exam}|A\}$.

$P(D^c|A) = 1 - P(A|D) = \frac{1}{451}$

$= \frac{450}{451}$

3
4. [30 points] Consider rolling two fair dice and let $X$ be the difference between the largest of
the two numbers shown and smallest of the two numbers shown.

(a) Obtain the probability mass function (pmf) of $X$.

(b) Determine $E[X]$, $E[X^2]$ and $Var(X)$.

\[
E[X] = 0 \left( \frac{6}{36} \right) + 1 \left( \frac{10}{36} \right) + 2 \left( \frac{8}{36} \right) + 3 \left( \frac{6}{36} \right) + 4 \left( \frac{4}{36} \right) + 5 \left( \frac{2}{36} \right) = \frac{70}{36} = \frac{35}{18}
\]

\[
E[X^2] = 0^2 \left( \frac{6}{36} \right) + 1^2 \left( \frac{10}{36} \right) + 2^2 \left( \frac{8}{36} \right) + 3^2 \left( \frac{6}{36} \right) + 4^2 \left( \frac{4}{36} \right) + 5^2 \left( \frac{2}{36} \right) = \frac{210}{36} = \frac{35}{6}
\]

\[
Var(X) = E[X^2] - (E[X])^2 = \frac{210}{36} - \left( \frac{70}{36} \right)^2 = \frac{660}{1296} = \frac{665}{1296}
\]

(c) Determine $E[-2X + 1]$, $E[-2X^2 + 1]$ and $Var(-2X + 1)$.

\[
E[-2X + 1] = -2E[X] + 1 = - \frac{70 + 18}{18} = - \frac{88}{18} = - \frac{26}{9}
\]

\[
E[-2X^2 + 1] = -2E[X^2] + 1 = - \frac{70 + 16}{6} = - \frac{86}{6} = - \frac{43}{3}
\]

\[
Var(-2X + 1) = (-2)^2 Var(X) = 4 \left( \frac{665}{1296} \right) = \frac{665}{324}
\]