University of Illinois Summer 2018

ECE 313: Exam I

Wednesday, June 27, 2018
10 - 10.50 a.m.
1013 ECEB

Name: (in **BLOCK CAPITALS**) Solutions

NetID: ____________________

Signature: ____________________

Instructions

This exam is printed **double-sided**, so make sure to look at all problems and both sides of every sheet.

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of **four** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

Grading	
1. 15 points	
2. 25 points	
3. 30 points	
4. 30 points	
Total (100 points)	

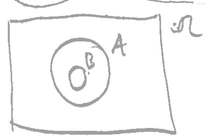
1. **[15 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score. For each of the following statements, determine if it is true or false for any three events A, B and C in a common probability space.

TRUE	FALSE	
☐	☒	$P(A\|B) + P(B) + P(C) \leq 1$.

e.g. $P(B) = 0.6 = P(C)$

☐	☒	$\underbrace{P(A\|B)}_{=1} + \underbrace{P(A^c\|B^c)}_{1} \leq 1$.

assume $\neq 0$



☐	☒	If A and B are mutually exclusive, then $P(AC) = P(A)P(C)$.

nothing to do with B.
If you thought it has A & C, it is still false.

☐	☒	If $E_1, \ldots, E_n$ is a partition of Ω, then $\sum_{i=1}^{n} P(B\|E_i) = P(B)$.

Law of total probability missing $P(E_i)$

☒	☐	If $E_1, \ldots, E_n$ is a partition of Ω, then $\sum_{i=1}^{n} P(E_i\|B)P(B) = P(B)$.

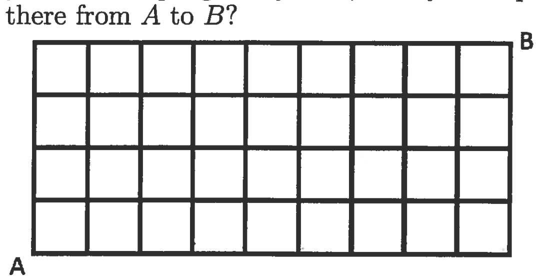
$= P(B) \sum_i P(E_i \mid B)$

can also use $P(E_i \mid B) P(B)$ $\rightarrow$
$= P(B \mid E_i) P(E_i)$

conditional probability must still have $P(\Omega \mid B) = 1$

2. **[25 points]** The two parts of this problem are unrelated.

(a) Consider the grid of points below. Suppose that starting at the point labeled A, you can can go one step up or one step to the right on the grid at each move. This continues until point B is reached. If you can't step up anymore, then you keep stepping right. If you can't step right anymore, then you keep stepping up. How many different paths are there from A to B?



Notice that we need 9 steps right and 4 steps up, but their order does not matter, so there are $\binom{9+4}{9}$ different paths

$$\binom{13}{9} = \frac{13!}{9!\,4!} = \frac{13(\cancel{12})11(\cancel{10})}{\cancel{4}(\cancel{3})\cancel{2}} = \boxed{715}$$

(b) Consider an 8x8 chess board and suppose you have 8 identical chess pieces. Determine the probability that none of the eight pieces share the same row, nor the same ~~row~~ column (i.e., there is only one piece in each row and in each column.)

All placements of the eight pieces are equally likely.

$|\Omega| = \binom{64}{8}$ because there are $8 \times 8 = 64$ spaces to put eight pieces.

Now, for our specific event, take the first column and choose one of its 8 rows to place a piece. Then, take the second row and choose of the 7 rows without a piece, on so forth. So there are $8!$ choices where pieces don't share rows nor columns.

So, $\boxed{8!/\binom{64}{8}} = \dfrac{8!\,8!\,56!}{64!}$

3. **[30 points]** Consider and ECE 313 student who is taking a final exam. If the student studied consistently during the semester, which happens with probability 0.3, then he/she has a 90% chance of getting an A in the course. If the student only studied during the last week of the semester, which happens with probability 0.6, then he/she has a 30% chance of getting an A in the course. If the student only studied the day before the exam, which happens with probability 0.1, then he/she has a 1% chance of getting an A in the course. Let A denote the event that the student got an A in the course.

(a) Determine $P(A)$.

Let $S = \{$studied whole semester$\}$
$W = \{$studied last week only$\}$
$D = \{$studied last day only$\}$

$$P(A) = P(A|S)P(S) + P(A|W)P(W) + P(A|D)P(D)$$
$$= 0.9(0.3) + 0.3(0.6) + 0.01(0.1)$$
$$= 0.27 + 0.18 + 0.001 = 0.451 = \boxed{\frac{451}{1000}}$$

(b) Determine $P\{\text{student studied only the day before the exam}|A\}$.

$$P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{0.01(0.1)}{0.451} = \frac{0.001}{0.451}$$
$$\boxed{= \frac{1}{451}}$$

(c) Determine $P\{\text{student studied more than just the day before the exam}|A\}$.

$$P(D^c|A) = 1 - P(D|A) = 1 - \frac{1}{451} = \boxed{\frac{450}{451}}$$

4. **[30 points]** Consider rolling two fair dice and let X be the difference between the largest of the two numbers shown and smallest of the two numbers shown.

(a) Obtain the probability mass function (pmf) of X.

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

$$\boxed{\begin{aligned} P_X(0) &= 6/36 = 1/6 \\ P_X(1) &= 10/36 = 5/18 \\ P_X(2) &= 8/36 = 2/9 \\ P_X(3) &= 6/36 = 1/6 \\ P_X(4) &= 4/36 = 1/9 \\ P_X(5) &= 2/36 = 1/18 \end{aligned}}$$

(b) Determine $E[X]$, $E\left[X^2\right]$ and $Var(X)$.

$$E[X] = 0\left(\frac{6}{36}\right) + 1\left(\frac{10}{36}\right) + 2\left(\frac{8}{36}\right) + 3\left(\frac{6}{36}\right) + 4\left(\frac{4}{36}\right) + 5\left(\frac{2}{36}\right) = \frac{70}{36} = \boxed{\frac{35}{18}}$$

$$E[X^2] = 0^2\left(\frac{6}{36}\right) + 1^2\left(\frac{10}{36}\right) + 2^2\left(\frac{8}{36}\right) + 3^2\left(\frac{6}{36}\right) + 4^2\left(\frac{4}{36}\right) + 5^2\left(\frac{2}{36}\right) = \frac{210}{36} = \boxed{\frac{35}{6}}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{210}{36} - \left(\frac{70}{36}\right)^2 = \frac{2660}{1296} = \boxed{\frac{665}{324}}$$

(c) Determine $E[-2X+1]$, $E\left[-2X^2+1\right]$ and $Var(-2X+1)$.

$$E[-2X+1] = -2E[X]+1 = \frac{-70+18}{18} = \boxed{\frac{-26}{9}}$$

$$E[-2X^2+1] = -2E[X^2]+1 = \frac{-70+6}{6} = \boxed{\frac{-32}{3}}$$

$$Var(-2X+1) = (-2)^2 Var(X) = 4\left(\frac{665}{324}\right) = \boxed{\frac{665}{81}}$$