ECE 313: Final Exam
Friday, August 4, 2017, 8-10 a.m.
ECEB 3017

Name: (in BLOCK CAPITALS) __________________________________________

NetID: ___________________________________________________________

Signature: ___________________________________________________________________________________________________

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 7 problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write \(\frac{3}{4}\) instead of \(\frac{24}{32}\) or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

<table>
<thead>
<tr>
<th>Grading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 14 points __________</td>
</tr>
<tr>
<td>2. 10 points __________</td>
</tr>
<tr>
<td>3. 14 points __________</td>
</tr>
<tr>
<td>4. 14 points __________</td>
</tr>
<tr>
<td>5. 25 points __________</td>
</tr>
<tr>
<td>6. 13 points __________</td>
</tr>
<tr>
<td>7. 10 points __________</td>
</tr>
<tr>
<td>Total (100 points) __________</td>
</tr>
</tbody>
</table>
1. [14 points] A message source that produces a sequence of 4 bits of information is equally likely to be in one of three states: $S_1$, $S_2$, and $S_3$. In state $S_k$, the 4 bit sequence can have at most $k$ 1's, with all such 4 bit sequences being equally likely. For example, in state $S_1$, the source can produce the sequences 0000, 1000, 0100, 0010, 0001 with each of sequences being produced with probability $\frac{1}{5}$.

Let $A$ be event that the sequence 1010 was produced by the source.

(a) [4 points] Find $P(A|S_2)$.

\[ S_2 \rightarrow a + \text{most} \geq 1 \text{ out of } 4 \]
\[ \Rightarrow |S_2| = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1 + 4 + 6 = 11 \]
\[ \Rightarrow P(A|S_2) = \frac{1}{11} \]

using reduced sample space to $S_2$.

(b) [6 points] Find $P(A)$.

Using total probability
\[ P(A) = P(A|S_1) P(S_1) + P(A|S_2) P(S_2) + P(A|S_3) P(S_3) \]
\[ = \frac{1}{11} \binom{4}{3} + \frac{1}{15} \binom{4}{1} \binom{3}{1} = \frac{26}{495} \]

(c) [4 points] Find the probability that the source was in state $S_2$ given event $A$.

\[ P(S_2|A) = \frac{P(A|S_2) P(S_2)}{P(A)} = \frac{\frac{1}{11} \binom{4}{3}}{\frac{26}{495}} = \frac{15}{26} \]
2. [10 points] Consider a binary hypotheses testing problem with observation $X$. Under $H_0$, $X \sim Binomial \left( 72, \frac{1}{3} \right)$, while under $H_1$, $X \sim Binomial \left( 72, \frac{2}{3} \right)$. It is known that $\pi_0 = 0.9$. £ $\Rightarrow$ $\pi_1 = 0.1$

(a) [6 points] Determine the MAP decision rule. Express the rule in terms of $k$ as simply as possible. In case of a tie in likelihoods, declare $H_1$ to be the hypothesis.

\[
\Delta(k) = \frac{P(k \mid H_1)}{P(k \mid H_0)} = \frac{\binom{72}{k} \left( \frac{2}{3} \right)^k \left( \frac{1}{3} \right)^{72-k}}{\binom{72}{k} \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^{72-k}} \geq \frac{\pi_0}{\pi_1} = 0.9 \Rightarrow \frac{2^k}{k!} \geq 0.9
\]

\[
2^k \geq 7.2 \geq 4 \Rightarrow k \geq 3.8
\]

\[
\begin{array}{c}
\text{\begin{align*}
\text{=0} & \text{ declare } H_1 \text{ if } k \geq 3.8 \\
\text{H}_0 & \text{ else}
\end{align*}}
\end{array}
\]

(b) [4 points] Express the approximate value of $P_{\text{false alarm}}$ for the MAP rule in terms of the $Q$ function, where $Q(c) = \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)du$. (To be definite, don’t use the continuity correction.)

\[
P_{\text{fa}} = P \left\{ \text{declare } H_1, H_0 \right\} = P \left\{ X \geq 3.8 \right\} = Q \left( \frac{3.8 - 2.4}{1.6} \right)
\]

\[
X \sim Binomial \left( 72, \frac{1}{3} \right) \text{ under } H_0 \Rightarrow \mu_X = \frac{72}{3} = 24
\]

\[
\sigma_X^2 = \frac{\mu(1-\mu)}{\frac{1}{3}} = 16
\]

\[
=0 \ P_{\text{fa}} = Q \left( \frac{\frac{3.8 - 2.4}{1.6}}{\frac{72}{2}} \right) = Q \left( \frac{7}{2} \right)
\]
3. [14 points] Let $c$ be a constant and $X$ be a random variable with pdf

$$f_X(u) = \begin{cases} 
c & u \in [-1, 1) \\
2c & u \in [1, 3] \\
0 & \text{else.}
\end{cases}$$

You may leave your answers to this problem in terms of $c$, except for part (a).

(a) [3 points] Obtain the value of the constant $c$.

$$\int_{-\infty}^{\infty} f_X(u) \, du = 1 = c \cdot \int_{-1}^{1} 2 \, du = 4c = 1$$

$$c = \frac{1}{4}$$

(b) [6 points] Determine the CDF $F_X(u)$ for all $u$.

$$F_X(u) = \begin{cases} 
0 & u < -1 \\
\frac{1}{2} & -1 \leq u < 1 \\
\frac{u+1}{2c} & 1 \leq u < 3 \\
1 & u \geq 3
\end{cases}$$

(c) [5 points] Obtain $E[\sin(X)]$. You may leave your answer in terms of $\sin$ and $\cos$.

$$E[\sin(X)] = \int_{-\infty}^{\infty} \sin(u) \, f_X(u) \, du = \int_{-1}^{1} \sin(u) \cdot 2c \, du + \int_{1}^{3} \sin(u) \cdot 2c \, du$$

Let $u = \cos\theta$,

$$= c \left[ -\cos \theta \right]_{1}^{3} - 2 \cos \theta \right]_{1}^{3}$$

$$= -c \left[ \cos(1) - \cos(3) + 2(\cos(3) - \cos(1)) \right]$$

$$= 2c \left[ \cos(3) - \cos(1) \right]$$
4. [14 points] Joe fishes with an unusual strategy: he waits for three fish to jump, then throws a net immediately after seeing the third fish jump. Fish jump according to a Poisson process with parameter $\lambda = 0.1$ jumps per minute. Assume that Joe begins waiting at time zero; the first fish jumps at time $T_1$, the second fish jumps at time $T_2$ and the third fish jumps at time $T_3$. You may leave powers of $e$ in your answer.

(a) [5 points] What’s the probability that $T_3 \geq 40$ minutes?

\[
P \left( T_3 > 40 \right) = P \left\{ N_{t=40} \leq 2 \right\} = P \left\{ N_{t=40} = 0 \right\} + P \left\{ N_{t=40} = 1 \right\} + P \left\{ N_{t=40} = 2 \right\}
\]

\[
= e^{-\lambda t} \frac{\lambda^0}{0!} + e^{-\lambda t} \frac{\lambda^1}{1!} + e^{-\lambda t} \frac{\lambda^2}{2!}
\]

\[
= e^{-0.1 \times 40} \frac{0.1^0}{0!} + e^{-0.1 \times 40} \frac{0.1^1}{1!} + e^{-0.1 \times 40} \frac{0.1^2}{2!}
\]

\[
= \left( 1 + 0.1 + \frac{0.01}{2} \right) e^{-0.1 \times 40}
\]

(b) [5 points] Suppose that Joe sees the first fish jump at time $T_1 = u > 0$ minutes and the second fish jump at time $T_2 = v > u > 0$ minutes. What is the conditional pdf $f_{T_3|T_1,T_2}(t|u,v)$ for all $t$?

\[
f_{T_3|T_1,T_2}(t|u,v) = \exp(-\lambda t) \quad \text{for } u < t < v
\]

\[
T_3 = v + \exp(-\lambda t)
\]

(c) [4 points] What is $E[T_3|T_1 = u, T_2 = v]$, for $0 < u < v$?

\[
E \left[ T_3 \mid T_1 = u, T_2 = v \right] = \frac{1}{\lambda} + v = 10 + v
\]
5. [25 points] Let $X$ and $Y$ be jointly continuous random variables with joint pdf
\[
    f_{X,Y}(u,v) = \begin{cases} 
        cv & v \in [0,1], \ u \in [v,2] \\
        0 & \text{else},
    \end{cases}
\]
for a constant $c$. You may leave this problem's answers in terms of $c$, except part (a).

(a) [3 points] Determine the value of the constant $c$.
\[
    \iint_{\mathbb{R}^2} f_{X,Y}(u,v) \, du \, dv = 1 = \int_0^1 \int_v^2 cv \, du \, dv = c \int_0^1 \int_v^2 du \, dv
\]
\[
    = c \frac{2}{3} \quad \Rightarrow \quad c = \frac{3}{2}
\]

(b) [4 points] Obtain the marginal pdf $f_Y(v)$ for all $v$.
\[
    f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v) \, du = \int_v^2 cv \, du = cv(2-v)
\]
\[
    \begin{cases} 
        cv(2-v) & v \in (0,1) \\
        0 & \text{else}
    \end{cases}
\]

(c) [4 points] Obtain the conditional pdf $f_{X|Y}(u|v)$ for all $u$ and $v$.
\[
    f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} = \begin{cases} 
        \text{undefined} & v \notin \text{int}(1) \\
        \frac{1}{2-v} & v \in \text{int}(1), \ u \in \text{int}(1)
    \end{cases}
\]
\[
    \frac{cv}{cv(2-v)} = \frac{1}{2-v}
\]
\[
    \because \ u \in \text{int}(v,2)
\]
(d) [8 points] Recall that

\[ f_{X,Y}(u,v) = \begin{cases} cv & v \in [0,1], u \in [v, 2] \\ 0 & \text{else} \end{cases} \]

Obtain the best unconstrained MMSE estimator of \( X \) from \( Y \), \( \hat{X} = g^*(Y) \). Notice that you are not estimating \( Y \) from \( X \).

Notice, either from (c) or from \( f_{X,Y} \), that conditional on \( Y = v \), \( X \sim \text{Uniform}(v, 2) \).

\[ g^*(Y) = E[X|Y] = \frac{Y + 2}{2} = \frac{1}{2} Y + 1 \]

(e) [6 points] Obtain the best linear MMSE estimator of \( X \) from \( Y \), \( \hat{X} = L^*(Y) = E[X|Y] \). Notice that you are not estimating \( Y \) from \( X \).

Notice from (d) that the best unconstrained estimator is linear, hence the best linear estimator must be the same line as (d):

\[ L^*(Y) = g^*(Y) = \frac{1}{2} Y + 1 \]

You could also do the whole calculation for \( L^*(Y) = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)} \cdot (Y - \mu_Y) + \mu_X \),

\[ \frac{19}{320} \quad \frac{-19}{640} \]
6. **[13 points]** Let $X$ and $Y$ be jointly uniform random variables with joint pdf

\[ f_{X,Y}(u,v) = \begin{cases} \frac{2}{3} & v \in [0,1], u \in [v,2] \\ 0 & \text{else} \end{cases} \]

(a) **[5 points]** Obtain $P\{Y - X^2 \leq 0\} = \int \int_{y \leq x^2} f_{X,Y}(u,v) \, du \, dv = \frac{2}{3} \int_0^1 \int_{v}^{\sqrt{v}} du \, dv = \frac{2}{3} \int_0^1 \frac{v}{2} \, dv = \frac{1}{3}$

(b) **[8 points]** Let $Z = \frac{1}{2}X - \frac{1}{3}Y$. Obtain the CDF $F_Z(s)$ for all $s$.

\[ F_Z(s) = \int_{-\infty}^{s} \int_{-\infty}^{\frac{s+Y}{2}} f_{X,Y}(u,v) \, du \, dv \]

For $s \in (-\infty, \frac{1}{2})$,

\[ F_Z(s) = \int_0^1 \int_{-s}^{\frac{s+Y}{2}} \frac{2}{3} \, du \, dv = \frac{1}{3} s \]

For $s \in \left[\frac{1}{2}, 1\right]$,

\[ F_Z(s) = 1 - \int_{s}^{\frac{2}{3}} \int_{\frac{2}{3}}^{\frac{2}{3}} \frac{2}{3} \, du \, dv = \frac{1}{3} \left(1 - \frac{2}{3}\right)^2 \]

For $s \in \left[1, \infty\right)$,

\[ F_Z(s) = 0 \]

For $s \in (0, \frac{1}{2})$,

\[ F_Z(s) = \begin{cases} 0 & s \leq 0 \\ \frac{4}{3} s & 0 < s < \frac{1}{2} \\ 1 - \frac{4}{3} (1-s)^2 & \frac{1}{2} \leq s \leq 1 \\ 1 & s > 1 \end{cases} \]
7. [10 points] (2 points per answer)
No partial credit. In order to discourage guessing, 2 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let $X$ and $Y$ be jointly random variables

**TRUE**

☐ **FALSE**

If $E[X + Y] = E[X] + E[Y]$ then $X$ and $Y$ are independent.

This always holds, even if not independent.

☐ **FALSE**

If $Var(X + Y) = Var(X) + Var(Y)$ then $X$ and $Y$ are independent.

This holds only if $X$ and $Y$ are just uncorrelated.

$$Var(X + Y) = Var(X) + Var(Y) + 2 Cov(X, Y)$$

☐ **FALSE**

If $f_{X,Y}(1, 2.5) = f_X(1)f_Y(2.5)$ then $X$ and $Y$ are independent.

This must hold everywhere on the plane, not just at one point.

$$f_{x,y}(x, y) = f_X(x)f_Y(y)$$

☐ **FALSE**

If $X$ and $Y$ are independent then $E[X^2 \sin(Y)] = E[X^2]E[\sin(Y)]$

If independent then $f_{X,Y}(x, y) = f_X(x)f_Y(y)$

$$E[X^2 \sin(Y)] = \iint x^2 \sin(y) f_{X,Y}(x, y) \, dx \, dy$$

$$= \int x^2 f_X(x) \, dx \int \sin(y) f_Y(y) \, dy$$

$$= E[X^2]^E[\sin(Y)]$$

☐ **FALSE**

If $E[Y|X] = aX + b$ then $E[X|Y] = \frac{1}{a}Y - \frac{1}{b}$

For example, if $X, Y$ are independent then $E[X|Y] = X$ and $E[Y|X] = Y$ doesn't hold.

$$E[Y|X] = a Y, \quad E[X|Y] = Y, \quad \neq$$