ECE 313: Exam II
Thursday, July 20, 2017
5:15 - 6:30 p.m.
3015 ECEB

Name: (in BLOCK CAPITALS) Solutions

NetID: ________________________________

Signature: ________________________________

Instructions

This exam is closed book and closed notes except that one 8.5" x 11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of five problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write \( \frac{3}{4} \) instead of \( \frac{24}{32} \) or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

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1. **[18 points]** Let $X$ be a random variable with CDF plotted below.

(a) Obtain $P\{X < 10\} = 1 - F_X(10)^- = 0.75$

(b) Obtain $P\{X > -5\} = 1 - F_X(-5) = 1 - 0.25 = 0.75$

(c) Obtain $P\{X = 1\} = F_X(1) - F_X(1^-) = 0$

   (no jump)

(d) Obtain $P\{|X| \leq 10\} = P\{-10 \leq X \leq 10\} = F_X(10) - F_X(-10^-)$

   $= 1 - 0.25 = 0.75$

(e) Determine if $X$ is a discrete, continuous or mixed-type random variable, and explain why.

   It is **mixed-type** because it has jumps and also strictly increasing and continuous portions.
2. [20 points] Let $X$ have pdf $f_X(u) = |u|$ for $u \in [-1, 1]$ and zero else.

(a) Determine $P\{X > -\frac{1}{2}\} = \int_{-\frac{1}{2}}^{1} f_X(u) \, du = \int_{-\frac{1}{2}}^{0} (-u) \, du + \int_{0}^{1} u \, du = \frac{5}{8}$

(b) Determine $P\{X < 0|X > -\frac{1}{2}\} = \frac{P\{X < 0, X > -\frac{1}{2}\}}{P\{X > -\frac{1}{2}\}} = \frac{P\{-\frac{1}{2} < X < c\}}{\frac{5}{8}}$

$= \frac{\frac{1}{5}}{\int_{-\frac{1}{2}}^{c} (-u) \, du = \frac{1}{5}}$

(c) Let $Z = X^3$, determine its pdf $f_Z(c)$ for all $c$, and determine its mean, $E[Z]$.

$f_Z(c) = P\{Z \leq c\} = P\{X^3 \leq c\} = P\{X \leq c^{\frac{1}{3}}\}$

$= f_X(c^{\frac{1}{3}})$

$= \frac{1}{3} c^{\frac{1}{3}} c^{-\frac{2}{3}} = \frac{1}{3} c^{\frac{1}{3}} \quad c \in [-1, 0]$  

$= \frac{1}{3} c^{-\frac{2}{3}} = \frac{1}{3} c^{\frac{-1}{3}} \quad c \in [0, 1]$  

$= 0 \quad \text{else}$

The pdf of $X$ is an even function with finite support, using l'Hôpital's rule: $E[X^3] = 0$
3. [20 points] The three parts of this problem are unrelated

(a) Let \( X \sim N(-2, 9) \). Determine \( P\{|X| < 3\} \) in terms of the \( \Phi \) or \( Q \) functions.

\[
\begin{align*}
\Pr\{|X| < 3\} &= \Pr\{-3 < X < 3\} = \Pr\left\{ \frac{-3 - (-2)}{\sqrt{9}} < \frac{X - (-2)}{\sqrt{9}} < \frac{3 - (-2)}{\sqrt{9}} \right\} \\
&= \Pr\left\{ -\frac{1}{3} < \frac{X + 2}{3} < \frac{5}{3} \right\} = \Phi\left(\frac{5}{3}\right) - \Phi\left(-\frac{1}{3}\right) = G\left(-\frac{1}{3}\right) - G\left(\frac{5}{3}\right)
\end{align*}
\]

(b) Let \( Z \sim N(\mu_Z, \sigma_Z^2) \). It is known that \( P\{Z < -1.34 \text{ or } Z > 3.34\} = 0.242 \) and that \( P\{Z > 3.34\} = 0.121 \). Determine \( \mu_Z \).

\[
\begin{align*}
P\{Z < -1.34 \text{ or } Z > 3.34\} &= P\{Z < -1.34\} + P\{Z > 3.34\} \\
&= 0.242 + 0.121 \\
&= 0.363
\end{align*}
\]

so \( \mu_Z \) is halfway between -1.34 and 3.34

\[
\mu_Z = \frac{-1.34 + 3.34}{2} = 1
\]

(c) Let \( Y \sim Binomial(100, 0.5) \). Use the Gaussian approximation with continuity correction to determine \( P\{Y < 60\} \) in terms of the \( \Phi \) or \( Q \) functions.

\[
\begin{align*}
\mu_Y = np &= 100 \times 0.5 = 50 \\
\sigma_Y^2 &= np(1-p) = 50 \times 0.5 = 25
\end{align*}
\]

\( Y \) is integer-valued, hence

\[
P\{Y < 60\} = \Pr\{Y = 59\} \approx \Pr\{59.5 \leq X \leq 60.5\}
\]

\[
\begin{align*}
\Pr\left\{ \frac{X - \mu_Y}{\sigma_Y} \leq \frac{59.5 - 50}{\sqrt{25}} \right\} &= \Phi\left(1.9\right)
\end{align*}
\]
4. [22 points] Suppose the number of visitors to a popular website follows a Poisson process with rate $4$ visitors per second. NOTE: you do not need to simplify the answers to this problem.

(a) What is the probability of exactly three visitors each minute in four consecutive minutes?

\[ P \left( N_1 - N_0 = 3, N_2 - N_1 = 3, N_3 - N_2 = 3, N_4 - N_3 = 3 \right) \]

This is because of independence of non-overlapping intervals.

\[ \sim \text{Poisson}(4 \cdot 60) \]

\[ = e^{-240} \frac{(240)^3}{3!} \]

(b) Let $0 < t_1 < t_2$, what is the probability that the first visitor arrives between $t_1$ and $t_2$? Assume that both $t_1$ and $t_2$ are measured in seconds.

Arrival of 1st visitor is $\text{Exp}(\lambda)$

\[ = \int_{t_1}^{t_2} P(t_1 < T < t_2) = F_{T_1}(t_2) - F_{T_1}(t_1) \]

\[ = e^{-\lambda t_2} - e^{-\lambda t_1} \]

(c) Let $0 < t_1 < t_2$, what is the probability that the second visitor arrives between $t_1$ and $t_2$ given that no visitors arrived before $t_1$? Assume that both $t_1$ and $t_2$ are measured in seconds.

Need at least 2 visitors between $t_1$ and $t_2$ for this to occur.

\[ = 1 - P \left( N_{t_2} - N_{t_1} < 2 \right) \]

\[ = 1 - \left[ e^{-\lambda (t_2 - t_1)} \left( \frac{\lambda (t_2 - t_1)}{1 + \lambda (t_2 - t_1)} \right)^0 + e^{-\lambda (t_2 - t_1)} \right] \]

\[ = 1 - e^{-\lambda (t_2 - t_1)} \left[ 1 + \lambda (t_2 - t_1) \right] \]
5. **[20 points]** Suppose under hypothesis $H_1$, $X$ has pdf $f_1(u) = |u|$ for $u \in [-1, 1]$ and zero else; while under hypothesis $H_0$, $X$ is uniform on $[-2, 2]$. Let $\pi_0 = \frac{5}{3}$.

(a) Obtain the MAP decision rule.

(b) Obtain $p_{false\ alarm}$ for the MAP rule.

(c) Determine the value(s) of $\pi_0$ for which $H_0$ would always be chosen.

From (a) need to compare $f_1(u) \leq \frac{\pi_0}{1-\pi_0} f_0(u)$

Highest point in $f_1(u)$ is at $u = \pm 1$.

$\Rightarrow f_1(1) \leq \frac{\pi_0}{1-\pi_0} f_0(1)$

$1 \leq \frac{\pi_0}{1-\pi_0} \left( \frac{1}{2} \right)$

$\Rightarrow \pi_0 > \frac{1}{2} \Rightarrow \frac{1}{2} \leq \pi_0 \

\Rightarrow \frac{5}{3} > \pi_0 \leq 1$