**University of Illinois** Summer 2016

# ECE 313: Final Exam

Saturday, August 6, 2016, 8-11 a.m.
ECEB 3020

**Name: (in BLOCK CAPITALS)** ____________________

**NetID:** ____________________

**Signature:** ____________________

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 10 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 18 points ________
2. 20 points ________
3. 18 points ________
4. 22 points ________
5. 22 points ________
6. 18 points ________
7. 18 points ________
8. 22 points ________
9. 12 points ________
10. 30 points ________

Total (200 points) ________

1. **[18 points]** Suppose under hypothesis $H_1$, $X$ has pdf $f_1(u) = \frac{1}{2} - \frac{1}{4}|u|$ for $u \in (-2, 2)$, but under hypothesis $H_0$, $X$ is uniformly distributed between $(-1, 1)$.

   (a) Let $\pi_0 = \frac{1}{3}$. Obtain the MAP decision rule.

   (b) Obtain $p_{false\ alarm}$ for the MAP rule from part $(a)$.

   (c) Determine all possible values of $\pi_0$ that cause the MAP rule to always choose $H_1$.

2. **[20 points]** Consider a Poisson process of rate $\lambda$.

(a) What is the probability that there are no arrivals in the interval $[0, 2]$?

(b) What is the probability that there are more than two arrivals in the interval $[0, 2]$?

(c) Given that there are two arrivals during $[0, 2]$, what is the probability that there is one arrival during $[0, 0.25]$?

(d) Obtain the probability that the second arrival occurs after a fixed time $t > 0$.

3. [**18 points**] Consider a two stage experiment. First, roll a die, with equiprobable sides labeled $1, 2, 3, 4, 4, 5$ (notice that 4 is on two sides of the die and 6 is not on the die). Let $X$ denote the number showing, and then flip a biased coin $X$ times, where tails shows $\frac{3}{4}$ of the time. Let $Y$ be the number of times tails shows.

(a) Obtain $P\{Y = 3|X = 4\}$.

(b) Obtain $P\{Y = 3\}$. Recall that 4 is on two sides of the die and 6 is not on the die.

(c) Obtain $P\{X = 4|Y = 3\}$.

4. **[22 points]** Let $X$ be a binomial random variable with parameters $n$ and $p$, and let $Y$ be be a binomial random variable with parameters $m$ and $q$. Assume that $X$ and $Y$ are independent.

   (a) Suppose (only for this part) that $n = 9$, $p = \frac{2}{3}$, $m = 16$ and that $q = \frac{1}{4}$. Obtain $E[2X + 3Y - 1]$ and $Var(2X + 3Y - 1)$.

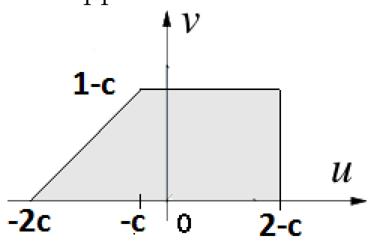
   (b) Obtain $Cov(2X, 3Y)$. You can leave your answer in terms of $n, p, m$ and $q$.

   (c) Obtain the joint pmf $p_{X,Y}(i, j)$ for all $i$ and $j$, and express it in terms of $p$ and $q$.

(d) Obtain the conditional pmf $p_{X|Y}(i|j)$ for all $i$ and $j$ in terms of $n, p, m$ and $q$.

(e) Suppose that $n = 6$ but you don't know $p$. You perform the experiment once and observe that $X^2 = 16$. Obtain the maximum likelihood estimate $\hat{p}_{ML}$.

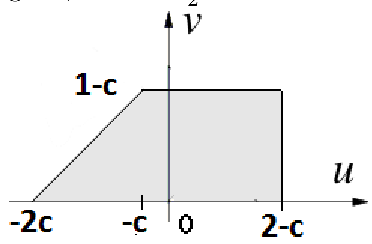
5. **[22 points]** Let $X$ and $Y$ be jointly uniform random variables with joint pdf $f_{X,Y}(u,v)$ with support in the shaded region below, where $c \geq 0$ is a constant.




(a) Let $c = \frac{1}{2}$. Obtain the joint pdf $f_{X,Y}(u,v)$ for all points in the $2-d$ plane.

(b) Again, let $c = \frac{1}{2}$. Obtain the marginal pdf $f_X(u)$ for all $u$.

(c) Again, let $c = \frac{1}{2}$. Obtain the conditional pdf $f_{Y|X}(v|u)$ for all $u$ and $v$.




(d) Find the value of the constant $c$ such that $X$ and $Y$ are independent.

6. [**18 points**] Suppose $X$ and $Y$ are jointly Gaussian random variables with $\mu_X = 3$, $\mu_Y = 0$, $\sigma_X^2 = 7$, $\sigma_Y^2 = 9$, and $Cov(X, Y) = 10$. Let $Z = X + Y$. NOTE: you can leave your answers for this problem in terms of the $Q$ function.

(a) Obtain $P(X < 1)$.

(b) Obtain $P(Z < 1)$.

(c) Obtain the best MMSE the linear estimator $\hat{E}[X|Z]$.

7. **[18 points]** A store sells two types of phones, model $A$ and model $I$. An $A$ phone battery drains after $X$ days, where $X \sim Exp(4)$. An $I$ phone battery drains after $Y$ days, where $Y \sim Exp(3)$. The random variables $X$ and $Y$ are independent.

   (a) Given that the battery in an $A$ phone has not drained in the the first $u$ days, what is the expected time before its battery drains?

   (b) Suppose that the $I$ phone is turned on after the $A$ phone's battery drains. Given that the battery in an $A$ phone has not drained in the the first $u$ days, what is the expected total time before both batteries drain?

   (c) What is the probability that an $A$ phone battery drains before an $I$ phone battery?

8. [**22 points**] Let $c$ be a constant and $X$ be a random variable with pdf.

$$f_X(u) = \begin{cases} \frac{1}{2} & u \in [-1, 0), \\ \frac{4}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases} .$$

You can leave your answers to this problem in terms of $c$, except for part $(a)$.

(a) Obtain the value of the constant $c$ in order for $f_X(u)$ to be a valid pdf.

(b) Determine the CDF $F_X(u)$ for all $u$. You can leave your answer in terms of the constant $c$.

(c) Obtain $E[X]$ and $E[X^3]$. Recall that $f_X(u) = \begin{cases} \frac{1}{2} & u \in [-1, 0), \\ \frac{4}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases}$.

(d) Let $Y = -\sqrt[3]{X}$. Obtain the CDF $F_Y(v)$ for all $v$.

9. [**12 points**] Suppose that an urn contains $g$ green balls and $r$ red balls. All balls are equally likely to be taken out of the urn.

   (a) Suppose that you grab a total of $k$ balls (no balls are put back). What is the probability of grabbing $x$ green balls?

   (b) Now suppose that all $k$ balls are returned to the urn, and this time you grab a total of $m$ balls. Let $A$ be the event that among the set of $m$ balls, exactly 2 green balls are included that were also grabbed the first time. Find $P(A)$.

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let $X$ be a random variable with an even function $f_X(u)$ as its pdf, i.e. $f_X(u) = f_X(-u)$ for all $u$. Assume $E[|X|^3]$ is finite.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $Var(X) = Var(|X|)$. |
| ☐ | ☐ | $E[(1-X)^2] = 1 + Var(X)$ |
| ☐ | ☐ | $Var[(1-X)^2] = Var(X)$ |
| ☐ | ☐ | $E[(1-X)^3] = 1 + 3Var(X)$. |

(b) Let $X$ and $Y$ be jointly random variables. $*$ denotes convolution.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $E[X+Y] = E[X] + E[Y]$ if and only if $X$ and $Y$ are uncorrelated. |
| ☐ | ☐ | $\mathrm{Var}(X+Y) = \sigma_X^2 + \sigma_Y^2$ if and only if $X$ and $Y$ are uncorrelated. |
| ☐ | ☐ | $f_{X+Y}(u) = f_X(u) * f_Y(u)$ if and only if $X$ and $Y$ are uncorrelated. |

(c) Suppose $X$ and $Y$ are jointly Gaussian random variables.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | If $\hat{E}[Y|X] = 3X + 1$, then $E[Y|X] = 3X + 1$. |
| ☐ | ☐ | If $\hat{E}[Y|X] = 3X + 1$ then $\hat{E}[X|Y] = \frac{1}{3}Y - \frac{1}{3}$. |
| ☐ | ☐ | If $E[Y|X]$ is constant, then $E[X|Y]$ is also constant. |