University of Illinois Summer 2016

ECE 313: Exam II

Thursday, July 21, 2016
8.45-10 p.m.
ECEB 1013

1. **[18 points]** Let A be a real-valued constant and consider the function

$$F_X(u) = \begin{cases} 0 & u < -2 \\ \frac{1}{2}u + 1 & -2 \le u \le -1 \\ \frac{1}{2} & -1 < u < 1 \\ 1 - Ae^{-u} & u \ge 1 \end{cases}.$$

(a) Find all values of the constant A that make this a valid CDF.
Solution: We need to make sure that the CDF is non-decreasing, which means that $\frac{1}{2} < 1 - Ae^{-u} \le 1$ for $u \ge 1$. This implies that $0 \le A \le \frac{1}{2}e$. All other properties of CDFs are satisfied.

(b) Obtain $P\{X \le -1\}$.
Solution: $P\{X \le -1\} = F_X(-1) = \frac{1}{2}(-1) + 1 = \frac{1}{2}$.

(c) Obtain $P\{X < 1\}$.
Solution: $P\{X < 1\} = F_X(1^-) = \frac{1}{2}$.

(d) Obtain $P\{X = -1\}$.
Solution: $P\{X = -1\} = F_X(-1) - F_X(-1^-) = 0$.

(e) Obtain $P\{|X| < 1\}$.
Solution: $P\{|X| < 1\} = P\{-1 < X < 1\} = F_X(1^-) - F_X(-1) = 0$.

2. **[28 points]** Let X be an exponential random variable with parameter 2.

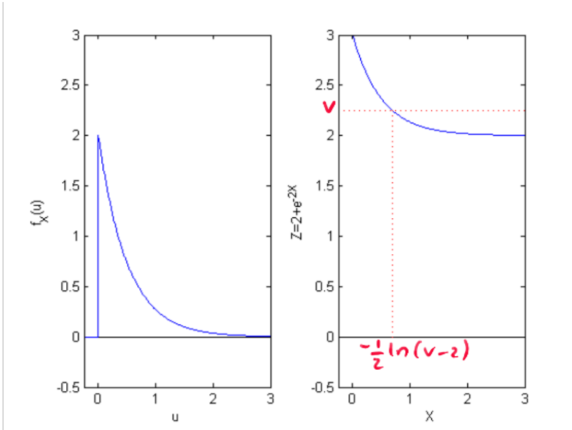
(a) Determine $P\{X > 5\}$.
Solution: $P\{X > 5\} = F_X^c(5) = e^{-2(5)} = e^{-10}$.

(b) Determine $P\{X < 8|X > 5\}$.
Solution: By the memoryless property, $P\{X < 8|X > 5\} = P\{X < 3\} = F_X(3) = 1 - e^{-2(3)} = 1 - e^{-6}$.

(c) Determine $E[X^2|X > 5]$.
Solution: As seen in exam 1, problem 3(d), by the memoryless property, $E[X^2|X > 5] = E[(Y+5)^2] = E[Y^2] + 10E[Y] + 5^2 = \frac{2}{2^2} + 10\frac{1}{2} + 25 = \frac{61}{2}$, where $Y \sim Exp(2)$.

(d) Recall that $X \sim Exp(2)$. Determine $E[e^{-2X}]$.
Solution: Using LOTUS: $E[e^{-2X}] = \int_{-\infty}^{\infty} e^{-2u} f_X(u) du = \int_0^{\infty} e^{-2u} \left(2e^{-2u}\right) du = 2\int_0^{\infty} e^{-4u} du = \frac{1}{2}$.

(e) Let $Z = e^{-2X} + 2$. Obtain the pdf of Z, $f_Z(v)$, for all v.
Solution: Consider the plots below. Clearly, $Z \in (2, 3)$, hence $F_Z(v) = 0$ for $v < 2$ and $F_Z(v) = 1$ for $v > 3$. For $2 < v < 3$, $F_Z(v) = P\{Z \le v\} = P\{e^{-2X} + 2 \le v\} = P\{X \ge -\frac{1}{2}\ln(v-2)\} = F_X^c\left(-\frac{1}{2}\ln(v-2)\right) = e^{-2\left(-\frac{1}{2}\ln(v-2)\right)} = v - 2$.



Taking derivatives we get $f_Z(v) = 1$ for $v \in (2,3)$ and zero else, i.e. $Z \sim Uniform(2,3)$.

(f) Determine $E[Z]$.

Solution: From part (e), $Z \sim Uniform(2,3)$, hence $E[Z] = \frac{2+3}{2} = \frac{5}{2}$.
One can also use the result from part (c) and linearlity of expectation.

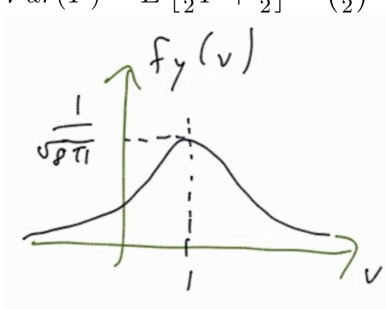
3. **[18 points]** Let X be Gaussian with mean -1 and variance 16.

(a) Express $P\{X^3 \leq -8\}$ in terms of the Φ function.

Solution: $P\{X^3 \leq -8\} = P\{X \leq -2\} = P\{\frac{X-(-1)}{\sqrt{16}} \leq \frac{-2+1}{4}\} = \Phi\left(-\frac{1}{4}\right)$, because $\frac{X-(-1)}{\sqrt{16}} \sim N(0,1)$.

(b) Let $Y = \frac{1}{2}X + \frac{3}{2}$. Sketch the pdf of Y, $f_y(v)$, clearly marking important points.

Solution: Y is a linear transformation of a Gaussian random random variable, X, so the shape of the pdf will be preserved, Y will also be Gaussian. This is why we can standardize Gaussians to use the Φ and Q functions. We need the mean and variance of Y to fully describe the pdf: $E[Y] = E\left[\frac{1}{2}X + \frac{3}{2}\right] = \frac{1}{2}E[X] + \frac{3}{2} = \frac{1}{2}(-1) + \frac{3}{2} = 1$, and $Var(Y) = E\left[\frac{1}{2}Y + \frac{3}{2}\right] = \left(\frac{1}{2}\right)^2 Var(X) = \frac{1}{4}(16) = 4$. Hence, $Y \sim N(1,4)$.



(c) Express $P\{Y \geq \frac{1}{2}\}$ in terms of the Φ function.

Solution: $P\{Y \geq \frac{1}{2}\} = P\{\frac{Y-1}{\sqrt{4}} \geq \frac{\frac{1}{2}-1}{2}\} = Q\left(-\frac{1}{4}\right) = \Phi\left(\frac{1}{4}\right) = 1 - \Phi\left(-\frac{1}{4}\right)$, because $\frac{Y-1}{\sqrt{4}} \sim N(0,1)$.
One could also do this directly using X.

4. **[20 points]** Suppose the number of fish detected by a remote underwater sensor follows a Poisson process with rate 5 fish per minute.

(a) What is the probability that exactly two fish are detected in a three minute period?

Solution: The number of fish detected in a three minute period has the Poisson distribution with mean $3\lambda = 3(5) = 15$, so the probability that the number is two is $\frac{e^{-15}(15)^2}{2!}$.

(b) What is the probability that exactly two fish are detected each minute in three consecutive minutes?

Solution: The number of fish detected in a one minute period has the Poisson distribution with mean $\lambda = 5$, so the probability that the number is two is $\frac{e^{-5}(5)^2}{2!}$, and because the number in each minute is independent of the number in the other minutes, then the probability that exactly two fish are detected each minute in three consecutive minutes is $\left(\frac{e^{-5}(5)^2}{2!}\right)^3 = \frac{e^{-15}(5)^6}{8}$.

(c) What is the probability that exactly two fish are detected the first minute given that exactly three fish are detected in the first four minutes?

Solution: As discussed in lecture, because the first minute is a subset of the first four minutes, and it's length is $\frac{1}{4}$ of the length of the four minute period, then the number of fish detected in the first minutes given that exactly three are detected in the first four minutes is $Binomial\left(3, \frac{1}{4}\right)$. Hence the probability that exactly two fish are detected the first minute given that exactly three are detected in the first four minutes is $\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = \frac{9}{64}$.

One can also use the definition of conditional probability directly to get the result.

(d) What is the expected value of the number of fish detected in the first minute given that exactly three fish are detected in the first four minutes?

Solution: As discussed in part (c), the number of fish detected in the first minute given that exactly three are detected in the first four minutes is $Binomial\left(3, \frac{1}{4}\right)$. Hence, the expected value of the number of fish detected the first minute given that exactly three are detected in the first four minutes is $3\left(\frac{1}{4}\right) = \frac{3}{4}$.

One could also use the approach from part (c) for all possible values of the number of fish detected in the first minute given that exactly three fish are detected in the first four minutes to get a pmf, and then use the definition of expected value.

5. **[16 points]** Suppose under hypothesis H_1, X has pdf $f_1(u) = e^{-2|u|}$ for all u, but under hypothesis H_0, X is exponentially distributed with parameter one. Let $\pi_0 = \frac{2}{3}$.

(a) Obtain the MAP decision rule.

Solution: The likelihood ratio is given by $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$. For $u < 0$, $f_0(u) = 0$, so H_1 will be chosen there. For $u > 0$, $\Lambda(u) = \frac{e^{-2u}}{e^{-u}} = e^{-u}$. The MAP rule compares the likelihood ratio to the threshold $\frac{\pi_0}{\pi_1} = 2$. We have $\Lambda(u) > 2$, which means $e^{-u} > 2$. This yields $u < -\ln(2)$, which means u is negative, but we are looking at $u > 0$, so that will never happen. Hence the MAP rule is the following: If $X < 0$, declare H_1 is true. If $X > 0$, declare H_0 is true.

(b) Obtain $p_{false\ alarm}$ for the MAP rule.

Solution: From part (a), the MAP rule is the following: If $X < 0$, declare H_1 is true. If $X > 0$, declare H_0 is true. We obtain

$$p_{false\ alarm} = P\{\text{declare } \ H_1|H_0\} = P\{X < 0|H_0\} = 0.$$

(c) Obtain p_{miss} for the MAP rule.

Solution: From part (a), the MAP rule is the following: If $X < 0$, declare H_1 is true. If $X > 0$, declare H_0 is true. We obtain

$$p_{miss} = P\{\text{declare } \ H_0|H_1\} = P\{X > 0|H_1\} = \frac{1}{2}.$$