University of Illinois Summer 2016

ECE 313: Exam I

Thursday, July 7, 2016
8.45 - 10 p.m.
1013 ECEB

1. **[16 points]** Consider three events A, B, and E such that: $P(A) = 0.3$, $P(AB) = 3P(ABE)$, $P(A \cup E) = 0.5$, $P(AB^c) = 0.03$, $P(BE) = 0.1$ and $P(A^cB^cE^c) = 0.48$.

   (a) Obtain $P(ABE)$.
   Solution: We are told that $P(AB) = 3P(ABE)$. Let $P(ABE) = x$, then $P(AB) = 3x$. Notice that $A = AB \cup AB^c$, which are two mutually exclusive events, hence $P(A) = P(AB \cup AB^c) = P(AB) + P(AB^c) = 3x + 0.03$ because we are told that $P(AB^c) = 0.03$, hence $x = \frac{P(A)-0.03}{3} = \frac{0.3-0.03}{3} = 0.09 = P(ABE)$.

   (b) Obtain $P(A^cBE^c)$.
   Solution: We are given $P(A \cup E) = 0.5$, hence $P\left((A \cup E)^c\right) = 1 - P(A \cup E) = 1 - 0.5 = 0.5$. If you look at the Karnaugh map, you'll see that $(A \cup E)^c = A^cB^cE^c \cup A^cBE^c$, which are two mutually excluive events, hence $P\left((A \cup E)^c\right) = P(A^cB^cE^c) + P(A^cBE^c)$. We are also told that $P(A^cB^cE^c) = 0.48$, hence $P(A^cBE^c) = 0.5 - 0.48 = 0.02$.

   (c) Are events AB and AE independent?
   Solution: For events AB and AE to be independent, we need $P(ABE) = P(AB)P(AE)$ because $ABAE = ABE$. In part (b) we obtained $P(ABE) = 0.09$, and we were told that $P(AB) = 3P(ABE) = 0.27$. Hence, we'd need $P(ABE) = 0.09 = P(AB)P(AE) = 0.27P(AE)$ which yields $P(AE) = \frac{0.09}{0.27} = \frac{1}{3}$. However, $P(AE) \leq P(A) = 0.3 < \frac{1}{3}$, which makes it impossible for $P(AE)$ to equal $\frac{1}{3}$. Therefore, events AB and AE not independent.

2. **[12 points]** The two parts of this problem are unrelated.

   (a) Consider a poker hand. Obtain $P\{\text{full house}\}$, where a full house consists of a three of a kind and a pair, that is three cards with the same rank and also two cards with the same rank but different than the rank of the other three cards.
   Solution: There are $\binom{13}{1}$ ways to select the rank for the three of a kind out of the 13 possible ranks. Then there ared $\binom{12}{1}$ ways to select the rank for the pair out of the remaninig 12 ranks (we need to remove the rank from the three of a kind). Once the rank of the three of a kind is selected, there are $\binom{4}{3}$ ways to select the the suits for those three cards out of the four possible suits. Also, once the rank of the pair is selected, there are $\binom{4}{2}$ ways to select the the suits for those two cards out of the four possible suits. Hence,

$$P\{\text{full house}\} = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{18}{51(49)5} = \frac{6}{4165} \approx 1.44 \times 10^{-3}.$$

   (b) Suppose that you have n distinct pairs of mittens. You put all inside a black bag and mix them, then you take out four mittens at random. How many ways are there to get at least one left mitten and two right mittens?
   Solution: Clearly, we need $n \geq 4$ to be able to take out four mittens. Having at least one left mitten and two right mittens leaves the choice of the fourth mitten to be either left or right. If the last mitten is left then there are two lefts and two rights, which can

be obtained in $\binom{n}{2}\binom{n}{2}$ ways because there are n left mittens to choose two from, and there are n right mittens to choose two from. On the ohter hand, if the last mitten is right then there one left and three rights, which can be obtained in $\binom{n}{1}\binom{n}{3}$ ways because there are n left mittens to choose one from, and there are n right mittens to choose three from. These two event are mutually exclusive because if we have only one left mitten we cannot have two. Hence, the total number of ways is the addition of both: $\binom{n}{2}\binom{n}{2} + \binom{n}{1}\binom{n}{3}$.

3. **[20 points]** Dilbert, Wally and Alice are tossing fair coins simultaneously but independently of each other. So they independently toss them at the same time and look at the three coins laying on the floor. Let X be the first time *at least* one of them gets a head on their simultaneous tosses.

   (a) Find the probability that X is strictly larger than 2.
   Solution: Notice that their tosses are independent of each others', so that each simultaeous toss has $P\{\text{at least 1 HEAD}\} = 1 - P\{\text{3 tails}\} = 1 - P\{\text{tails}\}P\{\text{tails}\}P\{\text{tails}\} = 1 - \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{7}{8}$. The simultaneous tossess are independent of each other,
   hence $X \sim Geometric(7/8)$. Therefore, $P\{X > 2\} = \frac{1}{8}^2 = \frac{1}{64}$.

   (b) Calculate $E[X]$.
   Solution: $X \sim Geometric(7/8)$, hence $E[X] = \frac{1}{\frac{7}{8}} = \frac{8}{7}$.

   (c) Find $E[X^2]$.
   Solution: $X \sim Geometric(7/8)$, hence $Var(X) = \frac{\frac{1}{8}}{\left(\frac{7}{8}\right)^2} = \frac{8}{49}$. Therefore $E[X^2] = Var(X) + \mu_X^2 = \frac{8}{49} + \left(\frac{8}{7}\right)^2 = \frac{72}{49}$.

   (d) Calculate $E[X^2|X > 2]$.
   Solution: Since X is a geometric random variable, we can use the memoryless property to argue that conditioned on $X > 2$ we can consider X to be equal to the sum of 2 and a geometric random variable (say Y) with parameter $\frac{7}{8}$. Then we can write $E[X^2|X > 2] = E[(2+Y)^2] = 4 + 4E[Y] + E[Y^2] = 4 + \frac{32}{7} + \frac{72}{49} = \frac{492}{49}$.

4. **[18 points]** A student is taking a multiple choice test with 10 questions. He did not study, so he answers each question by randomly choosing among the 5 possible answers, independently of any other questions. Let $X_i = 1$ if the student answers the i-th question correctly, and $X_i = 0$ if he doesn't. Let X be the total number of questions he answers correctly.

   (a) Obtain $P\{X = 3|X_1 = 1, X_{10} = 0\}$.
   Solution: The questions are independent, so if there are 3 correct answers total, and the first and last questions are correct and incorrect, respectively, then the other 2 correct answers must occur in the remaining 8 questions. This is a $Binomial\left(8, \frac{1}{5}\right)$ random variable, so $P\{X = 3|X_1 = 1, X_{10} = 0\} = \binom{8}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^6 = \frac{7(2)^{14}}{5^8}$.
   We could also do this directly using the definition of conditional probability:

$$P\{X = 3 \mid X_1 = 1, X_{10} = 0\} = \frac{P\{X=3, X_1=1, X_{10}=0\}}{P\{X_1=1, X_{10}=0\}} = \frac{P\{Y=2, X_1=1, X_{10}=0\}}{P\{X_1=1, X_{10}=0\}}$$
$$= \frac{P\{Y=2\}P\{X_1=1, X_{10}=0\}}{P\{X_1=1, X_{10}=0\}} = P\{Y = 2\} = \binom{8}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^6 = \frac{7(2)^{14}}{5^8}$$

   where $Y \sim Binomial\left(8, \frac{1}{5}\right)$ is the number of correcct answers in questions 2 through 9.

   (b) Obtain $P\{X_1 = 1|X_{10} = 0, X = 3\}$.
   Solution: Using the definition of contitional probability,

$$P\{X_1 = 1|X_{10} = 0, X = 3\} = \frac{P\{X_1=1, X_{10}=0, X=3\}}{P\{X_{10}=0, X=3\}} = \frac{P\{X_1=1, X_{10}=0, Y=3-1\}}{P\{X_{10}=0, Z=3\}}$$
$$= \frac{P\{X_1=1\}P\{X_{10}=0\}P\{Y=2\}}{P\{X_{10}=0\}P\{Z=3\}} = \frac{\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\binom{8}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{8-2}}{\left(\frac{4}{5}\right)\binom{9}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^6} = \frac{1}{3}$$

where $Z \sim Binomial\left(9, \frac{1}{5}\right)$ is the number of correcct answers in questions 1 through 9.

(c) If the student gets 2 points for each correct answer, obtain the mean and variance of the number of points he will get.

Solution: The number of correct answers he will get is $X \sim Binomial\left(10, \frac{1}{5}\right)$., which has mean $E[X] = 10\left(\frac{1}{5}\right) = 2$ and variance $Var(X) = 10\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = \frac{8}{5}$. Let Z be the total number of points he gets, then $Z = 2X$. By the scaling of mean and variance, $E[Z] = 2E[X] = 2(2) = 4$ and $Var(Z) = Var(2X) = 4Var(X) = 4\left(\frac{8}{5}\right) = \frac{32}{5}$.

5. **[16 points]** Let X be uniform on the set $\{n, n+1, \ldots, 2n\}$.

(a) Assume that $n = 3$. Obtain $E[X]$.

Solution: If $n = 3$, then $p_X(k) = \frac{1}{4}$ for $k \in \{3, 4, 5, 6\}$. Therefore, $E[X] = \sum_{u_i} u_i p_X(u_i) = \sum_{k=3}^{6} k\frac{1}{4} = \frac{1}{4}(3+4+5+6) = \frac{18}{4} = \frac{9}{2}$.

(b) Assume again that $n = 3$. Obtain $E\left[\frac{1}{X}\right]$.

Solution: If $n = 3$, then $p_X(k) = \frac{1}{4}$ for $k \in \{3, 4, 5, 6\}$. Therefore, by LOTUS, $E\left[\frac{1}{X}\right] = \sum_{u_i} \frac{1}{u_i} p_X(u_i) = \sum_{k=3}^{6} \frac{1}{k}\frac{1}{4} = \frac{1}{4}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = \frac{114}{480} = \frac{19}{80}$.

(c) Assume now that that n is unknown, but after performing the experiment, it is observed that $X = 5$. Obtain the maximum likelihood estimate of n, denoted by $\hat{n}_{ML}$.

Solution: We know that $p_X(k) = \frac{1}{n+1}$ for $k \in \{n, n+1, \ldots, 2n\}$, which decreases as n increases. Therefore, we want the smallest possible n. However, we need to make sure that the observed value of $X = 5$ is inside the set $\{n, n+1, \ldots, 2n\}$. So, we need $n \geq 5 \geq 2n$, which yields $\frac{5}{2} \geq n$. Hence, $\hat{n}_{ML} = 3$.

6. **[18 points]** Consider the binary hypothesis testing where X is uniform over the integers $\{1, 2, \ldots, 10\}$ under hypothesis H_0 and *Geometric*$(1/2)$ under hypothesis H_1.

(a) Derive the ML rule to decide on the underlying hypothesis.

Solution: The likelihood under H_0 is $p_X(k|H_0) = \frac{1}{10}$ for $k \in \{1, 2, \ldots, 10\}$ and zero else, while the likelihood under H_1 is $p_X(k|H_1) = \left(\frac{1}{2}\right)^k$ for integer $k > 0$. The ML rule chooses the largest one of the two, which is always $p_X(k|H_1) =$ for $k > 10$ because $p_X(k|H_0) = 0$ there. For $k \in \{1, 2, \ldots, 10\}$, $p_X(k|H_0) = \frac{1}{10} > p_X(k|H_1) = \left(\frac{1}{2}\right)^k$ is not true for $k \in \{1, 2, 3\}$, but it is true for $k \in \{4, 5, \ldots, 10\}$. Hence, the ML rules declares H_0 if $X \in \{4, 5, \ldots, 10\}$, and H_1 else (if $X \in \{1, 2, 3, 11, 12, \ldots\}$).

(b) What is the probability of false alarm for the ML rule?

Solution: $P_{\text{false alarme}} = P\{\text{declare } H_1|H_0\} = P(X \notin \{4, 5, \ldots, 10\}|H_0) = \sum_{k=1}^{3}\left(\frac{1}{10}\right) = 3\left(\frac{1}{10}\right) = \frac{3}{10}$.

(c) What is the probability of missed detection for the ML rule?

Solution: $P_{\text{miss}} = P\{\text{declare } H_0|H_1\} = P(X \in \{4, 5, \ldots, 10\}|H_1) = \sum_{k=4}^{10}\left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^{11}}{1-\left(\frac{1}{2}\right)} = \frac{127}{1024}$.