University of Illinois Summer 2015

ECE 313: Final Exam

Friday, August 7, 2015
8:00 a.m. — 11:00 a.m.

Name: **(in BLOCK CAPITALS)** ______________________________

NetID: ______________________________

Signature: ______________________________

Instructions

This exam is closed book and closed notes except that two 8.5”×11” sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 11 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 24 points ____________
2. 20 points ____________
3. 22 points ____________
4. 20 points ____________
5. 18 points ____________
6. 24 points ____________
7. 24 points ____________
8. 18 points ____________
9. 30 points ____________

Total (200 points) ____________

1. **[24 points]** For this problem, Dilbert flips a fair coin exactly 5 times each minute and Wally rolls multiple fair dice simultaneously multiple times.

   For each part below, **circle the letter of the correct answer and fill in the blanks for your selection.** If you circle more than one answer or if you do not circle an answer, you will receive zero credit.

   NOTE: The problem parts are independent.

   (a) Let M denote the number of heads that occur in the first 2 minutes. What type of random variable is M?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (b) Suppose Dilbert flipped his coin for five minutes without observing any heads. Let K denote the number of heads observed in the sixth minute. What type of random variable is K?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (c) Dilbert flips his coin until a heads occurs. Let Q denote the number of coin flips needed. What kind of random variable is Q?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (d) Suppose Dilbert flips the coin 7 times. Let H denote the number of heads that occur. What type of random variable is H?

A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

(e) Suppose Dilbert flips the coin 7 times and let H denote the number of heads that occur. After that, Wally rolls H dice simultaneously. Let T denote the number of twos he rolls. The conditional pmf $p_{T|H}$ has which of the following distributions?

A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

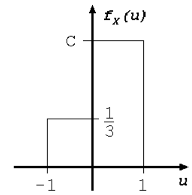
(f) Suppose Dilbert flips the coin 7 times and let H denote the number of heads that occur. After that, Wally rolls H dice simultaneously. Let T denote the number of twos he rolls. The random variable Y takes the value one if $T = H$, and zero otherwise. What kind of random variable is Y?

A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

2. [**20 points**] The random variable X has the pdf plotted here.



(a) Obtain the constant c for $f_X(u)$ to be a valid pdf.

(b) Obtain $E[6X+1]$ and $Var(6X+1)$

(c) Obtain the CDF of X, $F_X(u)$ for all u.

(d) Obtain $P\{X \geq \frac{1}{2}\}$

3. [**22 points**] Let X be an exponential random variable with parameter 2 and let Y be an exponential random variable with parameter 2, independent of X.

(a) Obtain the joint pdf of X and Y, $f_{X,Y}(u,v)$ for all u and v.

(b) Obtain the covariance between X and Y.

(c) Obtain the covariance between $2X+1$ and $3X-2$.

(d) Let $Z = \frac{2Y}{X+Y}$. Obtain the pdf of Z, $f_Z(c)$ for all c.

4. **[20 points]** Calvin is eating a box of 10 raisins by grabbing one at a time, tossing it in the air, and catching it with his mouth. If he doesn't catch it, then Hobbes eats it instead. Calvin catches the raisin in each attempt with probability $p \in (0, 1)$, independently of any other attempts. Let $X_i = 1$ if Calvin catches the raisin in his i-th attempt, and $X_i = 0$ if he doesn't. Let X be the total number of raisins he catches.

   (a) Obtain $P\{X = 3 | X_1 = 1, X_{10} = 0\}$.

   (b) Obtain the conditional pmf of X_1 given that $X_{10} = 0$ and $X = 3$, that is, obtain $P\{X_1 = k | X_{10} = 0, X = 3\}$, for all k.

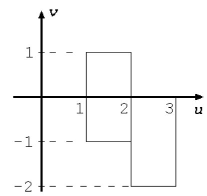
5. [**18 points**] Suppose that the eleven players in your soccer team go out to dinner and a movie. The team consists of one goalie, four defenders, four midfielders and two forwards.

   (a) At the movie theater, the eleven of you seat next to each other on a single row. What is the probability that the four defenders are seated next to each other?

   (b) After the movie, the eleven of you all go out to dinner and seat at a circular table. What is the probability that the four defenders are seated next to each other now? Note: rotated seatings are considered the same, but mirror images are not.

   (c) After dinner, the eleven of you all go get icecream and seat at a circular table. What is the probability that the two forwards are not seated next to each other in this circular table? Note: rotated seatings are considered the same, but mirror images are not.

6. [**24 points**] Let X and Y be jointy continuous random variables with joint pdf $f_{X,Y}(u,v) = ce^{-u}$ on the support set plotted below. Note: you can leave all of your answers for this problem in terms of c, without substituting its value, $c = \frac{1}{2(e^{-1}-e^{-3})}$.



(a) Obtain the marginal distribution of X, $f_X(u)$ for all u.

(b) Obtain the conditional distribution of Y given X, $f_{Y|X}(v|u)$ for all $(u,v) \in \mathbb{R}^2$.

(c) Obtain the best unconstrained estimator of Y from X, $\hat{Y} = g^*(X)$.

(d) Obtain $P\left\{Y \leq -\frac{1}{2}\right\}$.

7. **[24 points]** Let X and Y be jointly Gaussian random variables. The conditional distribution of Y given X is given by $f_{Y|X}(v|u) = \frac{1}{\sqrt{10\pi}} \exp\left(-\frac{(v-u-6)^2}{10}\right)$ for all $(u, v) \in \mathbb{R}^2$. It is known that the best constant estimator of Y from X is $\delta^* = 5$ and its corresponding minimum mean squared error $\text{MMSE}_{\delta^*} = 9$.

(a) Obtain the marginal pdf of Y, $f_Y(v)$ for all v.

(b) Obtain the best linear estimate of Y given that $X = 4$, $\hat{E}[Y|X = 4]$.

(c) Obtain the covariance between X and Y, $Cov(X, Y)$.

(d) Obtain the marginal pdf of X, $f_X(u)$ for all u.

8. [**18 points**] The two parts of this problem are unrelated.

(a) Let X be uniformly distributed on $(0, 1)$, and if $u \in (0, 1)$, then $f_{Y|X}(v|u) = \frac{1}{1-u}$ for $0 \leq v < 1 - u$, and zero else. Let $S = X + Y$. Obtain the pdf of S, $f_S(c)$.

(b) In this case, X and Y are independent random variables, each of which is uniformly distributed on $(0, 1)$, and let Z be a random variable such that $X + Y + Z = 1$. Clearly sketch the pdf of Z, $f_Z(w)$.

9. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A, B, C are three events such that $0 < P(A) < 1$, $0 < P(B) < 1$ and $0 < P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(A\|B) + P(A^c\|B) = 1$.
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	$P(A\|B) = P(B\|A)$, then $P(A) = P(B)$.

(b) Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively.

TRUE	FALSE	
☐	☐	$P_{FA}^{ML} + P_{MD}^{ML} = 1$.
☐	☐	$P_{FA}^{MAP} \leq P_{FA}^{ML}$.
☐	☐	$P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 \geq P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$.
☐	☐	If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.

(c) Suppose X, Y, Z are independent, Bernoulli($\frac{1}{2}$) random variables.

TRUE	FALSE	
☐	☐	$X + Y$ is independent of $X - Y$.
☐	☐	$\text{Cov}(X + 2Y, 2X - Y) = 0$.
☐	☐	$\text{Cov}(XY, XZ) = \frac{1}{8}$