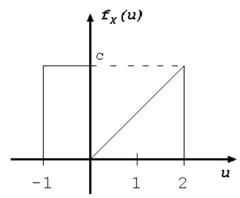
University of Illinois Summer 2015

ECE 313: Exam II

Thursday, July 23, 2015
4:00 p.m. — 5:15 p.m.
1013 ECEB

1. **[22 points]** Let X have the probability density function (pdf) $f_X(u)$ plotted below.



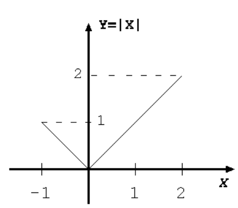
(a) Obtain the constant c for $f_X(u)$ to be a valid pdf.
Solution: The pdf has to be non-negative $c \geq 0$ and integrate to one $\left(\int_{-\infty}^{\infty} f_x(u)du = 1\right)$.
The integral is also the area under the curve so $(1)(c) + \frac{2(c)}{2} = 1$, which yields $c = \frac{1}{2}$.
If you want to do this via integration

$$1 = \int_{-\infty}^{\infty} f_x(u)du = \int_{-1}^{0} cdu + \int_{0}^{2} \frac{c}{2}udu = c + c = 2c$$

(b) Obtain the cumulative distribution function (CDF) of X, $F_X(u)$ for all u.
Solution: By definition, $F_X(u) = P\{X \leq u\} = \int_{-\infty}^{u} f_x(s)ds$.
If $u < -1$ there is nothing to integrate, so $F_X(u) = 0$.
If $-1 \leq u < 0$ we integrate the constant $c = \frac{1}{2}$ from -1 to u, so $F_X(u) = \int_{-1}^{u} \frac{1}{2}ds = \frac{u+1}{2}$.
If $0 \leq u \leq 1$ we integrate the constant $c = \frac{1}{2}$ from -1 to 0, which is equivalent to $F_X(0)$, and then we integrate the line $\frac{c}{2}u = \frac{u}{4}$ from 0 to u $F_X(u) = F_X(0) + \int_0^u \frac{s}{4}ds = \frac{1}{2} + \frac{u^2}{8}$.
Finally, if $u > 2$ there is nothing left integrate, so $F_X(u) = 1$.
Hence,

$$F_X(u) = \begin{cases} 0 & u < -1 \\ \frac{u+1}{2} & -1 \leq u < 0 \\ \frac{1}{2} + \frac{u^2}{8} & 0 \leq u \leq 2 \\ 1 & u > 2 \end{cases}$$

(c) Let $Y = |X|$, obtain the cumulative distribution function (CDF) of Y, $F_Y(v)$ for all v.
Solution: Following the steps described in class, we identify from the graph that the support of $f_X(u)$ as the interval $(-1, 2)$. The support of $f_Y(v)$ is then obtained by plotting $Y = |X|$ over the interval $(-1, 2)$ to get that the support of $f_X(u)$ as the interval $(0, 2)$.



By definition then, for $v \in (0, 2)$

$$F_Y(v) = P\{Y \leq v\} = P\{|X| \leq v\} = P\{-v \leq X \leq v\} = F_X(v) - F_X(-v)$$

Using the results from part (b), for $v \in (0,1)$, $F_X(v) - F_X(-v) = \frac{1}{2} + \frac{v^2}{8} - \frac{-v+1}{2} = \frac{1}{8}v^2 + \frac{1}{2}v$. However, notice that if $v \in (1,2)$ then $F_X(-v) = 0$, so $F_X(v) - F_X(-v) = \frac{1}{2} + \frac{v^2}{8}$. Therefore,

$$F_Y(v) = \begin{cases} 0 & v < 0 \\ \frac{1}{8}v^2 + \frac{1}{2}v & v \in [0,1] \\ \frac{1}{2} + \frac{v^2}{8} & v \in (1,2] \\ 1 & v > 2 \end{cases}$$

2. **[14 points]** Let $X \sim Uniform\left(0, \frac{1}{a}\right)$, that is, X has a uniform distribution on $\left(0, \frac{1}{a}\right)$.

   (a) If a is unknown, but it is observed that $X = \frac{1}{4}$, obtain the maximum likelihood estimate $\hat{a}_{ML}$ of a.
   Solution: The pdf of X is $f_X(u) = \frac{1}{\text{length of interval}} = \frac{1}{\frac{1}{a} - 0} = a$. To maximize this for the observed value $X = \frac{1}{4}$, we want to choose a as large as we can. However, as a increases, the range of X decreases, so we have to make sure it does not decrease to the point where the observation $X = \frac{1}{4}$ cannot occur (because it is not in the range). So, we need the right end of the range, $\frac{1}{a} \geq \frac{1}{4}$, which yields $\hat{a}_{ML} = 4$.

   (b) If $Y = 2X + 1$ and it is known that $Var(Y) = 12$, obtain a.
   Solution: By the scaling of variance: $12 = Var(Y) = Var(2X + 1) = 2^2 Var(X) = 4Var(X)$, which yields $Var(X) = 3$.
   The variance of a uniform random variable on (b, c) is $\frac{(c-b)^2}{12}$, so that $3 = Var(X) = \frac{(\frac{1}{a} - 0)^2}{12} = \frac{1}{12a^2}$, which yields $a = \pm\frac{1}{6}$, but because we need $a > 0$, then $a = \frac{1}{6}$.

3. **[25 points]** Bikes arrive at a certain intersection according to a Poisson process with arrival rate $\lambda = 2$ bikes/hour. Let N_t denote the number of bikes that arrive up until time t. Obtain:

   (a) $P\{$it takes more than 2 hours after the first bike arrives until the second bike arrives$\}$.
   Solution: The time between arrivals in a Poisson process with rate λ is an Exponential random variable with parameter λ, hence

   $$P\{\text{it takes more than 2 hours after the first bike arrives until the second bike arrives}\} \\ = P\{U_2 > 2\} = F^c_{U_2}(2) = e^{-\lambda(2)} = e^{-2(2)} = e^{-4},$$

   where $U_2 \sim Exp(2)$.
   One could also do this as follows:

   $$P\{\text{it takes more than 2 hours after the first bike arrives until the second bike arrives}\} \\ = P\{N_{t_1+2} - N_{t_1} = 0\} = \frac{e^{-\hat{\lambda}}\hat{\lambda}^0}{0!} = \frac{e^{-4}4^0}{0!} = e^{-4},$$

   where t_1 is the arrival time of the first bike, and the number of bike arrivals between time t_1 and $t_1 + 2$ is a Poisson random variable with parameter $\hat{\lambda} = 2(t_1 + 2 - t_1) = 4$, rate times length of interval.

   (b) $P\{N_3 - N_{1.5} = 2\}$.
   Solution: The number of bike arrivals between time 1.5 and 3 is a Poisson random variable with parameter $\hat{\lambda} = 2(3 - 1.5) = 3$, rate times length of interval. Hence

   $$P\{N_3 - N_{1.5} = 2\} = \frac{e^{-\hat{\lambda}}\hat{\lambda}^2}{2!} = \frac{e^{-3}3^2}{2!} = \frac{9}{2}e^{-3}$$

(c) $P\{N_3 = 8|N_1 = 3\}$.
Solution: We notice that the interval $(0, 3)$ overlaps with the interval $(0, 1)$, so N_3 and N_1 are not independent. However, if we split the interval $(0, 3) = (0, 1] \cup (1, 3)$, then $(0, 1)$ and $(1, 3)$ are non-overlapping, and hence $N_3 - N_1$ and N_1 are independent. There are 3 arrivals on $(0, 1)$, so the remaining 5 must occur on $(1, 3)$. Therefore,

$$P\{N_3 = 8|N_1 = 3\} = P\{N_3 - N_1 = 5\} = \frac{e^{-\hat{\lambda}}\hat{\lambda}^5}{5!} = \frac{e^{-4}4^5}{5!} = \frac{128}{15}e^{-4}$$

because $N_3 - N_1 \sim Poi(\hat{\lambda})$, where $\hat{\lambda} = 2(3 - 1) = 4$.
We can also do this directly using conditional probabilities:

$$\begin{aligned} P\{N_3 = 8|N_1 = 3\} &= \tfrac{P\{N_3=8,N_1=3\}}{P\{N_1=3\}} = \tfrac{P\{N_3-N_1=5,N_1=3\}}{P\{N_1=3\}} \\ &= \tfrac{P\{N_3-N_1=5\}P\{N_1=3\}}{P\{N_1=3\}} = P\{N_3 - N_1 = 5\} = \tfrac{128}{15}e^{-4} \end{aligned}$$

(d) Suppose λ is unknown but it is known that $P\{N_1 = 3|N_3 = 8\} = \binom{8}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5$, obtain the possible values of λ.
Solution: As discussed in class, if we condition on a certain number of arrivals within an interval, 8 arrivals on $(0, 3)$ in this case, then the time of each arrival is uniformly distributed on the interval. So, if we consider a subinterval that is $\frac{1}{3}$ the length of the original interval, $(0, 1)$ in this case, there is a $\frac{1}{3}$ chance of each arrival being on that subinterval. Hence, the number of arrivals in $(0, 1)$, conditioned on there being 8 arrivals on $(0, 3)$ is a binomial random variable with parameters $n = 8$ and $p = \frac{1}{3}$, independently of λ.
Therefore, any $\lambda > 0$ is possible.
You can also work this out by proceeding as in part (c) with a generic λ. You will realize that λ cancels out so that probability is independent of λ.

4. **[14 points]** Let $X \sim N(\mu, \sigma^2)$, that is, X is a Gaussian random variable with mean μ and variance σ^2. It is known that $P\{X > 2\} = 0.5$ and that $P\{X < 0.14\} = 0.2676$.

(a) Obtain μ and σ^2.
Solution: When we have a non-standard Gaussian, we will need to standardize it to use the tables.
To get the mean, we use the fact that

$$0.5 = P\{X > 2\} = P\{\frac{X - \mu}{\sigma} > \frac{2 - \mu}{\sigma}\} = Q\left(\frac{2 - \mu}{\sigma}\right).$$

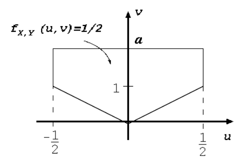
It is known, or from the tables, that $0.5 = \Phi(0)$, hence $0 = \frac{2-\mu}{\sigma}$, which yields $\mu = 2$. To get the variance, we use the fact that

$$0.2676 = P\{X < 0.14\} = P\{\frac{X - \mu}{\sigma} < \frac{0.14 - \mu}{\sigma}\} = \Phi\left(\frac{-1.86}{\sigma}\right) = Q\left(\frac{1.86}{\sigma}\right).$$

From the tables $0.2676 = Q(0.62)$, hence $0.62 = \frac{1.86}{\sigma}$, which yields $\sigma = 3$, and hence $\sigma^2 = 9$.

(b) Find constants a and b such that $Y = aX + b$ is a Gaussian random variable with mean 1 and variance $\frac{1}{4}$, that is, for $Y \sim N\left(1, \frac{1}{4}\right)$.
Solution: From part (a) we know that $X \sim N(3, 9)$.
From scaling of variance, we know that $\frac{1}{4} = Var(Y) = Var(aX+b) = a^2Var(X) = 9a^2$.
Hence $a = \pm\frac{1}{6}$.
From scaling of expectation, we know that $1 = E[Y] = E[aX + b] = aE[X] + b = 2a + b$.
If $a = \frac{1}{6}$ then $b = \frac{2}{3}$. If $a = -\frac{1}{6}$ then $b = \frac{4}{3}$.

5. **[25 points]** Let X and Y be jointly continuous random variables with joint probability density function (pdf) $f_{X,Y}(u,v) = \frac{1}{2}$ in the area plotted below, and zero else. Obtain:



(a) The constant a for $f_{X,Y}(u,v)$ to be a valid joint pdf.
Solution: For a valid joint pdf, it has to integrate to one over the whole 2-d plane $\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dvdu = 1\right)$.
In this case, we can just look at the area of support because the joint pdf is constant, hence

$$1 = \frac{1}{2}\left[2\left(\frac{\frac{1}{2}(1)}{2}\right) + (1)(a-1)\right],$$

which yields $a = \frac{5}{2}$.

(b) The marginal probability density function (pdf) of X, $f_X(u)$ for all u.
Solution: To obtain the marginal $f_X(u)$, we need to integrate out v for a fixed u, so $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv$. Notice that because the joint pdf is constant, for a fixed u we can simply take the length of the interval(s) in v over which the joint pdf is non-zero and multiply it by the constant value of the joint pdf, $\frac{1}{2}$.
If $u \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$ then there is nothing to integrate, and $f_X(u) = 0$.
If $u \in \left(-\frac{1}{2}, 0\right)$, the joint pdf is nonzero only for $v \in (-2u, a)$, so $f_X(u) = \frac{1}{2}(a - (-2u)) = \frac{1}{2}(a + 2u)$.
If $u \in \left(0, \frac{1}{2}\right)$, the joint pdf is nonzero only for $v \in (2u, a)$, so $f_X(u) = \frac{1}{2}(a - 2u)$.
If you want to do this via the integrals, then if $u \in \left(-\frac{1}{2}, 0\right)$, $f_X(u) = \int_{-2u}^{a} \frac{1}{2}dv = \frac{1}{2}(a+2u)$.
If $u \in \left(0, \frac{1}{2}\right)$, $f_X(u) = \int_{2u}^{a} \frac{1}{2}dv = \frac{1}{2}(a - 2u)$.
Hence,

$$f_X(u) = \begin{cases} 0 & u \notin \left(-\frac{1}{2}, \frac{1}{2}\right) \\ \frac{1}{2}(a+2u) = \frac{1}{2}\left(\frac{5}{2} + 2u\right) & u \in \left(-\frac{1}{2}, 0\right) \\ \frac{1}{2}(a-2u) = \frac{1}{2}\left(\frac{5}{2} - 2u\right) & u \in \left(0, \frac{1}{2}\right) \end{cases}$$

(c) $E[X]$.
Solution: With the joint pdf being constant, the mean over X would partition the area of support of the joint pdf into two equal areas, which occurs, by inspection, at $u = 0$. Hence $E[X] = 0$.
We can also do this via the definition of expectation:

$$E[X] = \int_{-\infty}^{\infty} u f_X(u)du = \int_{-\frac{1}{2}}^{0} u\frac{1}{2}(a+2u)du + \int_{0}^{\frac{1}{2}} u\frac{1}{2}(a-2u)du = 0.$$

(d) The conditional pdf of Y given X, $f_{Y|X}(v|u)$ for all u, v.
Solution: Recall that if $f_X(u) \neq 0$, then $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$. We know that $f_{X,Y}(u,v)$ is constant $\left(\frac{1}{2}\right)$ over its support. From part (c) we know that $f_X(u)$ takes

different functional forms on $u \in \left(-\frac{1}{2}, 0\right)$ and on $u \in \left(0, \frac{1}{2}\right)$, and that the range of v in those intervals also changes. Hence

$$f_{Y|X}(v|u) = \begin{cases} \text{undefined} & u \notin \left(-\frac{1}{2}, \frac{1}{2}\right) \\ \frac{1}{a+2u} = \frac{2}{5+4u} & u \in \left(-\frac{1}{2}, 0\right), v \in (-2u, a) \Rightarrow Y|X = u \text{ is } Uniform\left(-2u, \frac{5}{2}\right) \\ \frac{1}{a-2u} = \frac{2}{5-4u} & u \in \left(0, \frac{1}{2}\right), v \in (2u, a) \Rightarrow Y|X = u \text{ is } Uniform\left(2u, \frac{5}{2}\right) \\ 0 & \text{else} \end{cases}$$