University of Illinois Summer 2014

ECE 313: Exam I

Monday, July 7, 2014
7:00 p.m. — 8:15 p.m.
269 Everitt Lab

1. **[12 points] Guessing Restaurant Quality:** Suppose you go to a restaurant and do not know the restaurant quality beforehand. Assume the restaurant quality level is distributed over $\{1, 2, 3, 4, 5\}$ with equal probability, and if the quality level is i, then the number of people dining in the restaurant follows Poisson distribution with mean $10i$. Numerical answer is not needed for this problem.

   (a) (6 points) What is the probability that you see 40 people dining? Be as explicit as possible.

   (b) (6 points) What is the probability that the restaurant quality level is at least 3 given that you see 40 people dining? Be as explicit as possible.

2. **[25 points] Testing Bern(p) versus Bern(q):** Consider a hypothesis testing problem in which the observation X follows the Bernoulli distribution with mean p under H_0 and the Bernoulli distribution with mean q under H_1. Assume $p = 0.4$ and $q = 0.6$

   (a) (5 points) Find the ML decision rule. Be as explicit as possible.

   (b) (5 points) Find the $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

   (c) (5 points) Suppose $P(H_0) = \frac{1}{4}$ and $P(H_1) = \frac{3}{4}$. Find the MAP decision rule. Be as explicit as possible.

   (d) (5 points) Continue part (c). Find the average probability error p_e for the MAP rule. Be as explicit as possible.

   (e) (5 points) Find all possible values of $P(H_0)$ such that the $p_{\text{false alarm}}$ for the MAP rule is zero, assuming the tie is broken in favor of H_1.

3. **[16 points] Verifying Polynomial Identities:** Suppose there are two polynomials $F(x)$ and $G(x)$. Assume that the maximum degree, or the largest exponent of x, in $F(x)$ and $G(x)$ is d. Consider the following approach for verifying whether $F(x)$ is identical to $G(x)$. We choose an integer n uniformly at random in the range $\{1, 2, \ldots, m\}$. Then we compute the values $F(n)$ and $G(n)$. If $F(n) = G(n)$, we decide that $F(x)$ is identical to $G(x)$; if $F(n) \neq G(n)$, we decide that $F(x)$ is not identical to $G(x)$.

   (a) (8 points) Find a non-trivial upper bound on the probability that we made an error in terms of d and m. Hint: A polynomial of degree up to d has at most d roots.

   (b) (8 points) Suppose we choose r integers $n_1, \ldots, n_r$ uniformly and independently at random in the range $\{1, 2, \ldots, m\}$. If $F(n_i) = G(n_i)$ for all $i = 1, \ldots, r$, we decide that $F(x)$ is identical to $G(x)$; if $F(n_i) \neq G(n_i)$ for some $1 \leq i \leq r$, we decide that $F(x)$ is not identical to $G(x)$. Find a non-trivial upper bound on the probability that we made an error in terms of d, m and r.

4. **[29 points] Balls and Bins:** Suppose n balls are thrown into m bins independently and uniformly at random, where $n = \alpha m$ for a positive integer $\alpha \geq 1$ and $m \geq 2$ is an integer.

   (a) (5 points) Let X be the number of balls in the first bin. What is the distribution of X in the limit as $m \to \infty$ (Write down the probability mass function)? Be as explicit as possible.

(b) (6 points) Continue part (a). Find the fraction of empty bins on average exactly as well as using the distribution of X in the limit as $m \to \infty$.

(c) (6 points) What is the *exact* probability that all bins contain α balls? Be as explicit as possible.

(d) (6 points) Suppose all bins contain exactly α balls. Each round we pick one ball out of all possible balls uniformly and independently at random and then place it back. What is the probability of seeing at least one ball from every bin after m rounds? Be as explicit as possible.

(e) (6 points) Continue part (d). What is the expected number of rounds to see at least one ball from every bin? Be as explicit as possible.

5. **[18 points] Estimating π:** Consider the following approach for estimating π. Suppose there is 2×2 square region S in the plane centered at $(0, 0)$, i.e., $S = \{(x, y) : x \in [-1, 1], y \in [-1, 1]\}$. A circle C of radius 1 centered at $(0, 0)$ lies in the square and has area π, i.e., $C = \{(x, y) : x^2 + y^2 \leq 1\}$. We throw n needles in the square and assume each needle is independently and uniformly distributed in the square. Let X denote the number of needles lying in the circle.

(a) (6 points) Find the probability mass function of X. Be as explicit as possible.

(b) (6 points) Suppose $X = k$ where $1 \leq k \leq n$ is an integer. Derive the maximum likelihood estimator of π. Show your derivation.

(c) (6 points) We use Chebychev inequality to decide the confidence interval. If π is to be estimated within 0.1 (i.e., the half-width of the confidence interval is 0.1) with 96% confidence level, how many needles should be thrown?