**University of Illinois** Summer 2013

# ECE 313: Exam III

**Name: (in BLOCK CAPITALS)** ______________________________

**Net ID:** ____________________

**University ID Number:** ____________________

**Signature:** ______________________________

## Instructions

This exam is closed book and closed notes except that three 8.5"×11" page of notes is permitted. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

Write your answers in the boxes provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75). It is OK for your final answers to include terms like $\binom{100}{20}$, $\binom{a}{10}$, $a^k - b^{k-1}$, $2^{50}$, and so on.

SHOW YOUR WORK. answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page.

1. 20 points ____________
2. 15 points ____________
3. 15 points ____________
4. 15 points ____________
5. 10 points ____________

Total (75 points) ____________

1. **[20 points]** Consider a random variables $X$ and $Y$ with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & 0 < x \leq 1, 0 < y \leq 1 \\ A, & -1 \leq x \leq 0, -1 \leq y \leq 0. \end{cases}$$

(a) **[5 points]** Find the value of $A$

| |
|---|
| $A =$ |

(b) **[5 points]** Find the marginal pdf of $Y$.

| |
|---|
| $f_Y(y) =$ |

**PROBLEM 1, CONTINUED**

(c) [**5 points**] Find the conditional pdf $f_{X|Y}(x|y)$.

| |
|---|
| $f_{X\|Y}(x\|y) =$ |

(d) [**5 points**] Find $P(X^2 + Y^2 \leq 1)$

| |
|---|
| $ANS =$ |

2. **[15 points]** Suppose arrival of calls to a call center can be modeled as a Poisson process with arrival rate $\lambda$ calls per minute.

   (a) **[5 points]** Derive the maximum likelihood estimate $\hat{\lambda}_{ML}$ if $k$ calls are received in a $T$ minute interval, as a function of $k, T$.

   | $\hat{\lambda}_{ML} =$ |
   |---|

   (b) **[5 points]** Given $\lambda = 2$, what is the conditional probability that 3 calls arrive in the **first two minutes** given that 2 calls arrive in the **second** minute?

   | $ANS =$ |
   |---|

(c) [**5 points**] Given $\lambda = 2$, what is the conditional probability that 3 calls arrive in the **second minute** given that 5 calls arrive in the **first three** minutes?

| |
|---|
| $ANS =$ |

3. **[15 points]** Let $X$ be an uniform random variable over the interval $[-2, 1]$, and let $Y = e^{X^2}$

(a) **[7 points]** Find the support of pdf of $Y$: $f_Y(y)$.

| $ANS =$ |
| --- |

(b) **[8 points]** Determine the pdf of $Y$ on its support

| $f_Y(y) =$ |
| --- |

4. **[15 points]** Consider the following binary hypothesis testing problem. Under hypothesis $H_1$, $X$ is a standard Normal random variable with mean 0 and variance 1, that is,

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Under hypothesis $H_0$, $X$ is uniform over the interval $\left[-\sqrt{\frac{\pi}{2}}e, \sqrt{\frac{\pi}{2}}e\right]$, that is,

$$f_0(x) = \begin{cases} \frac{1}{(\sqrt{2\pi})e} & , x \in \left[-\sqrt{\frac{\pi}{2}}e, \sqrt{\frac{\pi}{2}}e\right] \\ 0 & , else \end{cases}$$

Determine the following. (Hint: sketch both pdfs; your solutions must be in terms of $\pi$ and $e$. For your convinience, $\sqrt{\frac{\pi}{2}}e \approx 3.4069$)

(a) **[5 points]** Find the ML decision rule.

| |
|---|
| $ANS:$ |

**PROBLEM 4, CONTINUED**

(b) **[5 points]** Find $p_{miss}$ for the ML decision rule in terms of the $Q$ function with positive arguments.

| ANS : |
|---|

(c) **[5 points]** Let $\pi_0 = P(H_0), \pi_1 = P(H_1)$. Find the minimum value of $\frac{\pi_1}{\pi_0}$ such that MAP always declares $H_1$ to be true.

| $ANS =$ |
|---|

5. **[10 points]** An airline sold 100 tickets for a plane with 70 seats. Each passenger actually shows up at the airport with probability $\frac{1}{2}$. Using Guassian approximation with continuity correction, what is the probability that there are not enough seats on the plane for all passengers to show up? Write your answer in terms of the $Q$ function.

(a) $ANS =$