Solution for midterm II

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Problem. 1 Solution:

1) The area below the pdf is $2c$, so $c = 1/2$.
2) $E[X] =$ the center of mass $= 3/2$.
3) $F_X(t) = \begin{cases} 0, t \leq 0. \\ ct * t/2 = t^2/4, 0 < t \leq 1 \\ 1/4 + c(t-1) = t/2 - 1/4, 1 < t \leq 2 \\ 1 - (3-t)c(3-t)/2 = 1 - (t-3)^2/4 = -t^2/4 + 3t/2 - 5/4, 2 < t \leq 3 \\ 1, t > 3. \end{cases}$

Problem. 2 Solution:

1) X can take values in $\{0, 1, 2, 3\}$.
2) $P(X = 3) = (1-p)^7$.As long as $L1 - L7$ are all good.
3) Let $q = 1-p$, $P(X = 0) = p(1-q^3)(1-q^2)^2 + q(1-q^3)(p^2+p^2-p^4) = 7/16$, if $p = 1/2$.

Problem. 3 Solution:

1) $P(X = 3) = \frac{\binom{r}{3}\binom{52-r}{1}4!}{52*51*50*49} = (52-r)\frac{\binom{r}{3}}{\binom{52}{4}}, r = 3, 4, 5, ...52.$
2) $P(X = 1) = \frac{r\binom{52-r}{3}}{\binom{52}{4}}$. Maximize it

$$\frac{r\binom{52-r}{3}}{(r-1)\binom{53-r}{3}} = \frac{r}{r-1}\frac{50-r}{53-r} \geq 1 \Rightarrow r \leq 53/4 = 13.25.$$

So $\hat{r}_{ML} = 13$ if $X = 1$ is observed.

Problem. 4 Solution:

1) $Y|H_1 \sim Geo(0.9)$, $Y|H_0 \sim Geo(0.1)$.

$$\Lambda(k) = \frac{P(Y=k|H_1)}{P(Y=k|H_0)} = (1/9)^{k-2} \text{ compare with } 1 \Rightarrow \begin{cases} choose\ H_1, k=1 \\ choose\ H_0, k \geq 3 \\ either\ is\ OK, k=2. \end{cases}$$

2)

$$\Lambda(k) = \frac{P(Y=k|H_1)}{P(Y=k|H_0)} = (1/9)^{k-2} \text{ compare with } \pi_0/\pi_1 = 1/3 \Rightarrow k \text{ compare with } 2.5 \Rightarrow$$

$$\begin{cases} choose\ H_1, k=1,2 \\ choose\ H_0, k \geq 3 \end{cases}$$

3)

$$p_{false\ alarm} = P(Y \leq 2|H_0) = P(Geo(0.1) \leq 2) = 1 - 0.9^2 = 0.19$$

$$p_{miss} = P(Y \geq 3|H_1) = P(Geo(0.9) \geq 3) = 0.1^2 = 0.01.$$

Problem. 5 Solution:

1) $F_X(c) = P(X \leq c) = \begin{cases} 0, c < 0 \\ c/2 + \frac{1}{2}(1-e^{-c}) = 1/2 + c/2 - e^{-c}/2, 0 \leq c < 1 \\ \frac{1}{2} + \frac{1}{2}(1-e^{-c}), 1 \leq c \end{cases}$

2) $E[X] = \frac{1}{2}E\big[Unif[0,1]\big] + \frac{1}{2}E\big[Exp(1)\big] = \frac{1}{4} + \frac{1}{2} = \frac{3}{4.}$

3) $E[X^2] = \frac{1}{2}E\big[Unif[0,1]^2\big] + \frac{1}{2}E\big[Exp(1)^2\big] = \frac{1}{2}\left\{Var\left(Unif[0,1]\right) + 1/2^2 + Var\left(Exp(1)\right) + 1^2\right\} = \frac{1}{2}\left\{1/12 + 1/4 + 1 + 1\right\} = 7/6$. $Var(X) = E[X^2] - E[X]^2 = 7/6 - 9/16 = 29/48$

Problem. 6 Solution:

1) S_2 is the same as the sum of numbers of two balls chosen from 6 balls, so $P(S_2 = 6) = \frac{|\{(1,5),(2,4)\}|}{\binom{6}{2}} = 4/30 = 2/15$.

2) S_4 is the same as the sum of numbers of four balls chosen from all 6 balls. Let X_i be the number of the i-th ball in the 4 balls, then $S_4 = X_1 + X_2 + X_3 + X_4$, and $E[X_i] = (1+6)/2 = 7/2$, so $E[S_4] = 7/2 * 4 = 14$..

3) $P(A, S_2 = 6) = \frac{1}{6}2/\binom{5}{2} = 1/30$. So $P(A|S_2 = 6) = \frac{P(A,S_2=6)}{P(S_2=6)} = \frac{1/30}{2/15} = 1/4$.