University of Illinois Summer 2013

ECE 313: Midterm II

Name: (in BLOCK CAPITALS) ______________________________________

Net ID: _________________________

University ID Number: _________________________

Signature: ______________________________________

Instructions

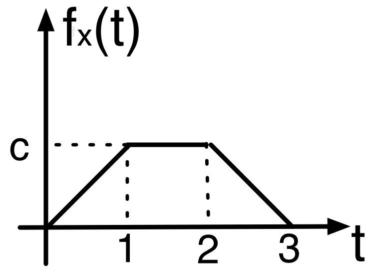
This exam is closed book and closed notes except that two 8.5"×11" page of notes is permitted: both sides of a sheet is allowed. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

Write your answers in the boxes provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75). It is OK for your final answers to include terms like $\binom{100}{20}$, $\binom{a}{10}$, $a^k - b^{k-1}$, 2^{50}, and so on.

SHOW YOUR WORK. answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page.

1. 20 points	__________
2. 20 points	__________
3. 20 points	__________
4. 20 points	__________
5. 20 points	__________
6. 25 points	__________
Total (125 points)	__________

1. **[20 points]** Suppose the pdf of a continuous random variable X is shown below



(a) [**5 points**] Find c

$c =$

(b) [**5 points**] Find $E[X]$.

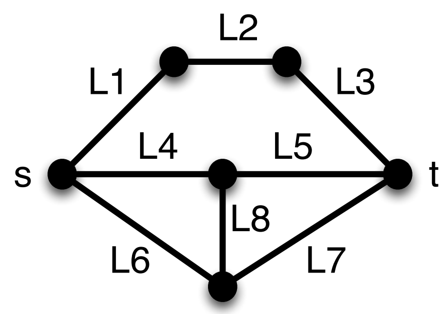
$ANS =$

PROBLEM 1, CONTINUED

(c) **[10 points]** Find the CDF of X and sketch it. Clearly mark important points and values. Use full and empty circles to indicate the value of the CDF at discontinuities.

$F_X(t) =$

2. **[20 points]** In the following network, all links L_i have capacity 1. Each link may fail independently of others with probability $p \in (0, 1)$. The source is node s and the destination is node t . Let the capacity of the network be denoted by X.



(a) **[3 points]** Find all possible values of X.

$ANS =$

(b) **[5 points]** Find $P(X = 3)$, express it as a function of p

$ANS =$

PROBLEM 2, CONTINUED

(c) **[12 points]** Suppose $p = 1/2$, find $P(X = 0)$.

$ANS =$

3. **[20 points]** A standard deck of 52 cards contains 4 aces. Suppose we shuffle all cards randomly so that all 52! permutations being equally likely. Let X be the number of aces in the toppest r cards ($1 \leq r \leq 52$).

   (a) **[8 points]** Find $P(X = 3)$, as a function of r.

$ANS =$

   (b) **[12 points]** Find the ML estimator of r if $X = 1$ is observed.

$ANS =$

4. **[20 points]** Alice has a unfair coin which gives Head w.p 0.9 and Tail w.p. 0.1. Alice randomly chooses one side of the coin, and flips the coin until she sees her chosen side. Let Y be the number of times Alice flips the coin. That is

   - H_1 : Alice chooses Head as her side, and flips the coin until Head first appears.
   - H_0 : Alice chooses Tail as her side, and flips the coin until Tail first appears.

   And Y is the number of times Alice flips the coin.

   (a) **[6 points]** Describe the ML decision rule, given that you observed $Y = k$ for $k = 1, 2, 3, ...$ Express it as directly in terms of k as possible.

ANS :

   (b) **[6 points]** Suppose Alice chooses Head w.p 3/4 and chooses Tail w.p 1/4, that is, $\pi_0 = 1/4, \pi_1 = 3/4$. Describe the MAP decision rule given that $Y = k, k = 1, 2, 3...$, express it as directly in terms of k as possible.

ANS :

PROBLEM 4, CONTINUED

(c) **[8 points]** Find $p_{false\ alarm}$ and p_{miss} for the MAP rule in (b).

$p_{false\ alarm} =$

$p_{miss} =$

5. **[20 points]** The random variable X is defined as follows: A fair coin is tossed. If Head shows, we randomly and uniformly choose the value of X from the interval $[0, 1]$; if Tail shows, X is an Exponential random variable with prameter 1. That is,

   - If Head shows, $X \sim Uniform[0, 1]$.
   - If Tail shows, $X \sim Exponential(1)$.

   (a) **[8 points]** Find CDF of X, express it as a piecewise function of c

   $F_X(c) =$

   (b) **[6 points]**Find $E[X]$. Hint: You can use the law of total probability for expectation.

   ANS =

PROBLEM 5, CONTINUED

(c) [**6 points**]Find $Var(X)$.

ANS =

6. **[25 points]** 6 balls, numbered 1 through 6, are in a bag. Suppose we do the following two-step experiment:

   1) randomly pick a ball from the bag and throw it away.
   2) draw balls randomly from the bag one by one without replacement, let S_i be the sum of numbers on the first i balls.

   For example, if the ball numbered 1 is thrown away and the rest balls are drawn as an order of $(2, 3, 4, 5, 6)$, then $S_1 = 2, S_2 = 5, S_3 = 9, S_4 = 14, S_5 = 20$.

   (a) **[8 points]** Find $P(S_2 = 6)$.

ANS =

   (b) **[8 points]** Find $E[S_4]$.

ANS =

(c) **[9 points]** Let $A = \{\text{the number of the ball thrown away is } 6\}$. Find $P(A|S_2 = 6)$.

ANS =