Solution for midterm I

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Problem. 1 Solution:

1) true. Because $P(A \cup B) = P(A) + P(B) - P(AB) \leq P(A) + P(B)$.
2) false. It works only when A and B are disjoint events.
3) true. A is independent of any combination of B and C.
4) false. $E[X + Y^2] = E[X] + E[Y^2]$.
5) true. $\sum_{i=1}^{\infty}(e-1)e^{-i} = (e-1)\frac{e^{-1}}{1-e^{-1}} = 1$.
6) false. $Var(2X+1) = 4Var(X)$.
7) false. $P(A^c)+P(B^c) \geq P(A^c \cup B^c) = 1-P(AB)$, so $P(A^c)+P(B^c)+P(AB) \geq 1$, but $1-\frac{3}{4}+1-\frac{2}{3}+\frac{1}{3} < 1$.
8) true. $A \setminus (B \cap C^c) = A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (A \cap B^c) \cup (A \cap C) = (A \setminus B) \cup (A \setminus C^c)$
9) false. $p_X(k) = (1-p)^{k-1}p$ obtains its maximum at $k = 1$.
10) false $P(AB) = P(A)P(B|A) = 0.15, P(AB^c) = 0.15, P(A^cB) = 0.55, P(A^cB^c) = 0.15$

Problem. 2 Solution:

1)

$$P[\text{all top } 4 \text{ cards are aces}] = \frac{1}{\binom{52}{4}} = \frac{4!}{52*51*50*49} = 3.6938 \times 10^{-6}$$

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2)

$$P[\text{none of top } 4 \text{ cards are aces}] = \frac{\binom{52-4}{4}}{\binom{52}{4}} = \frac{48*47*46*45}{52*51*50*49} = 0.71874.$$

3)

$$P[\text{none of top } 4 \text{ cards are aces}|\text{the top card is SPADE}] = \frac{\binom{13-1}{1}\binom{52-1-4}{3}}{\binom{13}{1}\binom{52-1}{3}} = \frac{12*47*46*45}{13*51*50*49} = 0.71874$$

Problem. 3 Solution:

1) $P(X \leq 3) = (3/6)^5 = 1/32$.
2) $P(X = 4) = P(X \leq 4) - P(X \leq 3) = (4/6)^5 - (3/6)^5 = (2/3)^5 - (1/2)^5 = \frac{781}{6^5}$.
3) $ANS = \binom{5}{2}(2/6)^2(4/6)^3 = \binom{5}{2}\frac{4^4}{6^5} = \binom{5}{2}\frac{8}{3^5} = \frac{80}{3^5}$

Problem. 4 Solution:

1)

	international	not international
male	$300 - 60 = 240$	$2000 * 0.6 - 240 = 960$
female	$300 * 20\% = 60$	$2000 * 0.4 - 60 = 740$

$$P[\text{the student is male}|\text{the student is not international}] = \frac{960}{960 + 740} = \frac{960}{2000 - 300} = \frac{960}{1700} = \frac{48}{85}$$

2)

$$P[\text{the student is international}|\text{the student is female}] = \frac{60}{60 + 740} = \frac{60}{800} = \frac{3}{40}$$

3) false. The two are not independent. Because given the student is international, the probability for it to be male increases.

Problem. 5 Solution:

1) $P(M = 3) = \frac{3!}{6^3} = \frac{1}{36}$
2) $E[M] = E[Geo(3/6) + Geo(2/6) + Geo(1/6)] = 6/3 + 6/2 + 6/1 = 11$.
3) $P(M = 4) = \frac{3*3*3!+3*3*2}{6^4} = \frac{1}{18}$. Count the number of outcomes satisfying $M = 4$: the 4-th value has to be one of $2, 4, 6$. If there is one value of $1, 3, 5$ in the first 3 trials, the number is: (3 ways to choose a value from $1, 3, 5$) $\times$ (3 ways to choose the position for the value chosen from $1, 3, 5$) $\times$ (3! ways to order $2, 4, 6$ in the rest 3 positions); if there is no value from $1, 3, 5$ in the first 3 trials, the number is: (3 ways to choose two positions from the first 3 positions so that to put two same values) $\times$ (3 ways to choose a value from $2, 4, 6$ and put it in the two positions chosen) $\times$ (2! ways to choose the other 2 values from $2, 4, 6$ and put them in the rest two positions). And, at last, the total number of outcomes is 6^4 because each position can have 6 values.