Lecture 26 (already)

In the previous sections, we studied the distribution of a single random variable. If two or more random variables are closely interconnected, how can we express the distribution of them, together?

Competitor

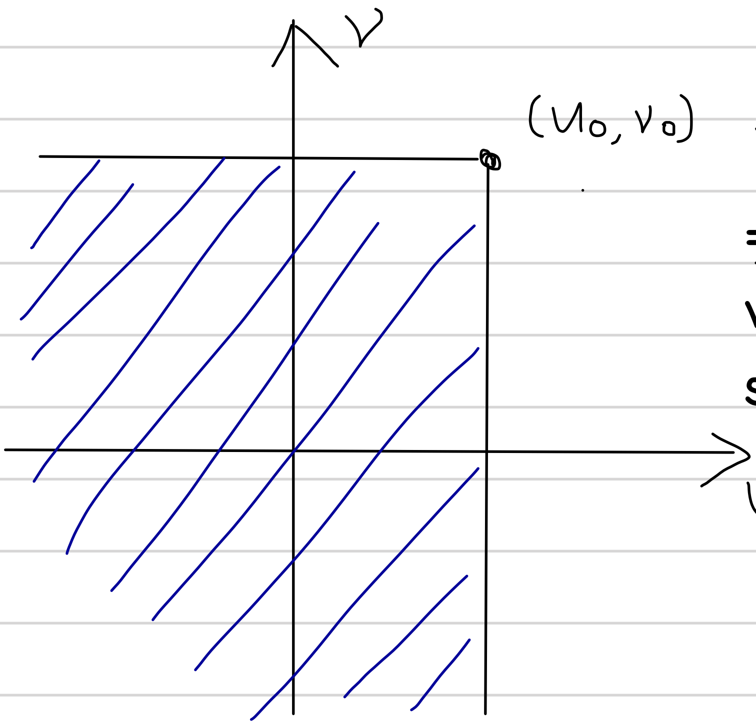
ex) Google Android ⟷ Apple iOS
Google search — Apple iPhone

partner

X: Google Stock Price
Y: Apple Stock Price
$\Rightarrow$ **dependent**

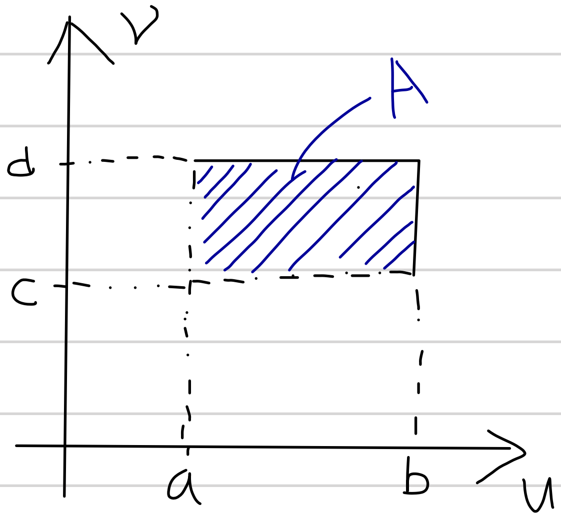
4.1 Joint Cumulative Distribution Function (CDF)

Def) $F_{X,Y}(u_0, v_0) = P(X \le u_0, Y \le v_0)$



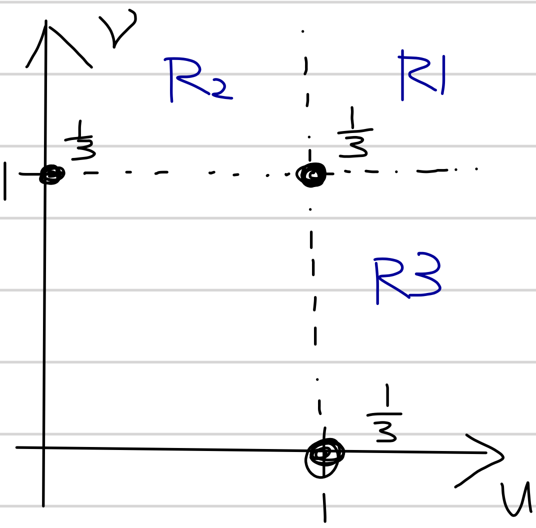
$F_{XY}(u_0, v_0)$
= Probability that a random vector (X, Y) falls in the shaded region.

Ex) Probability that (X, Y) falls in the shaded region?



$$P[(X,Y)\in A] = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c)$$

Ex) Find $F_{XY}(u,v)$ if $P(X=u, Y=v) = \begin{cases} \frac{1}{3} & \text{if } u=1 \text{ or } v=1 \\ 0 & \text{otherwise} \end{cases}$



(X,Y) = one of the three points with prob $\frac{1}{3}$.

Let $R1 = \{(u,v) \mid u \geq 1, v \geq 1\}$
$R2 = \{(u,v) \mid 0 \leq u < 1, v \geq 1\}$
$R3 = \{(u,v) \mid u \geq 1, 0 \leq v < 1\}$

↓ : note!

$$\begin{cases} \text{For } (u,v) \text{ in } R1, & F_{XY}(u,v) = 1 \\ \text{For } (u,v) \text{ in } R2, & F_{XY}(u,v) = \frac{1}{3} \\ \text{For } (u,v) \text{ in } R3, & F_{XY}(u,v) = \frac{1}{3} \\ \text{For } (u,v) \notin R_1 \cup R_2 \cup R_3, & F_{XY}(u,v) = 0 \end{cases}$$

Properties of $F_{XY}(u,v)$

1) $0 \le F_{XY}(u,v) \le 1$ (Since Fxy is a probability)
2) Non-decreasing in u and v, respectively (Since CDF)
3) Right Continuous in u and v, respectively
4) $\lim_{u\to\infty} \lim_{v\to\infty} F_{XY}(u,v) = 1$

$\lim_{u\to-\infty} F_{XY}(u,v) = 0$, $\lim_{v\to\infty} F_{XY}(u,v) = 0$

Q) How can we find the CDF of X (or Y) from Fxy?

$$F_X(u) = P(X \le u) = P(X \le u, Y \le \infty) = \lim_{v\to\infty} F_{XY}(u,v)$$

Similarily, $F_Y(v) = \lim_{u\to\infty} F_{XY}(u,v)$.

4.2 Joint Probability Mass Function (Joint PMF)

For discrete-type r.v.s X and Y, the joint PMF is

Def) Pxy (u, v)= P{X=u, Y=v}

Marginal PMF

$$p_X(u) = \sum_v p_{XY}(u,v), \quad p_Y(v) = \sum_u p_{XY}(u,v)$$

Conditional PMF

$$p_{Y|X}(v|u_0) = P[Y=v \mid X=u_0] = \frac{p_{XY}(u_0,v)}{p_X(u_0)}$$ (Defined only when $p_X(u_0) > 0$)

$$p_{X|Y}(u|v_0) = P[X=u \mid Y=v_0] = p_{XY}(u,v_0) / p_Y(v)$$

Def) Support of $p_X(u) \triangleq \{u \in \mathbb{R} \mid p_X(u) > 0\}$

$\Rightarrow$ Conditional pmf $p_{Y|X}(v|u_0)$ is well-defined only for u_0 in the support of f_X.

Ex) Table 4.1: A simple joint pmf.

$Y=3$	0.1	0.1	
$Y=2$		0.2	0.2
$Y=1$		0.3	0.1
	$X=1$	$X=2$	$X=3$

Q1) Find $p_X(u)$?

$p_X(1) = 0.1$, $P_X(2) = 0.1 + 0.2 + 0.3 = 0.6$, $P_X(3) = 0.1 + 0.2 = 0.3$

$p_X(u) = 0$ otherwise.

Q2) Find $P_Y(v)$?

$P_Y(1) = 0.3 + 0.1 = 0.4$, $P_Y(2) = 0.2 + 0.2 = 0.4$, $P_Y(3) = 0.1 + 0.1 = 0.2$

Q3) $$P_{Y|X}(v|2) = \begin{cases} 3/6 & \text{if } v=1 \\ 2/6 & \text{if } v=2 \\ 1/6 & \text{if } v=3 \\ 0 & \text{otherwise} \end{cases}$$

Q4) Support of $f_X(u)$? A: $\{1, 2, 3\}$

Properties of PMF.

1) $0 \le p_X(u) \le 1$ (Why? it is a probability)
2) $\sum_{(u,v) \in \mathbb{Z}^2} P_{XY}(u,v) = 1$.

EX) X= # showing on a die
Y = # of heads after a coin is tossed X times,

Q1) joint pmf of X and Y?

$$p_{XY}(i,j) = P(X=i, Y=j) = P(Y=j \mid X=i)\,P(X=i)$$

$$= \begin{cases} \binom{i}{j}\left(\frac{1}{2}\right)^{i} \cdot \frac{1}{6} & \text{if } \quad 0 \le i \le j \le 6. \\ 0 & \text{otherwise} \end{cases}$$

Q2) Conditional pmfs?

$$p_{Y|X}(j \mid i) = \binom{i}{j} 2^{-i} \qquad \sim \text{Binomial}\left(i, \tfrac{1}{2}\right)$$

$$p_{X|Y}(i \mid j) = \frac{p_{XY}(i,j)}{p_Y(j)} = \frac{\frac{1}{6}\binom{i}{j}2^{-i}}{\sum_{i=1}^{6}\frac{1}{6}\binom{i}{j}2^{-i}} \qquad \text{for } 0 \le j \le 6.$$

Lecture 27

4.3. Joint Probability Density Function (Joint PDF)

Def) The random variables X and Y are jointly continuous if there exists a function $f_{X,Y}(u,v)$ such that

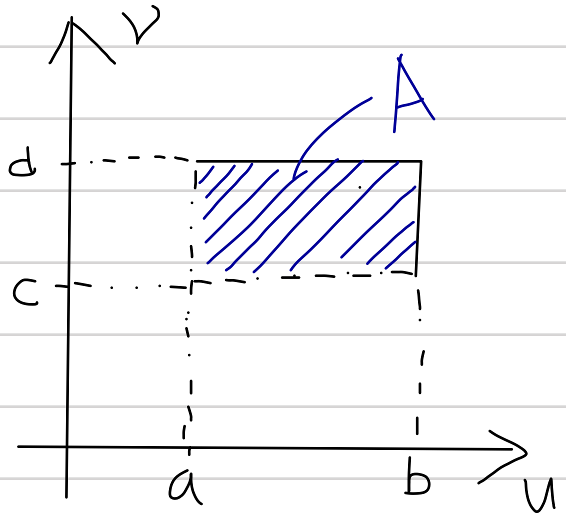
$$F_{XY}(u_0,v_0)=\int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{XY}(u,v)\,dv\,du \quad \text{for } \forall u_0, v_0 \in \mathbb{R}$$

$f_{X,Y}(u, v)$ is called the joint probability density function of X and Y.

Ex) $P\{(X,Y)\in A\}$

$$=\int_a^b\int_c^d f_{XY}(u,v)\,dv\,du$$

= Volume of the cake over the region!



$\rightarrow P\{(X,Y)=(u_0,v_0)\} = 0$ (Why? X and Y are continuous)

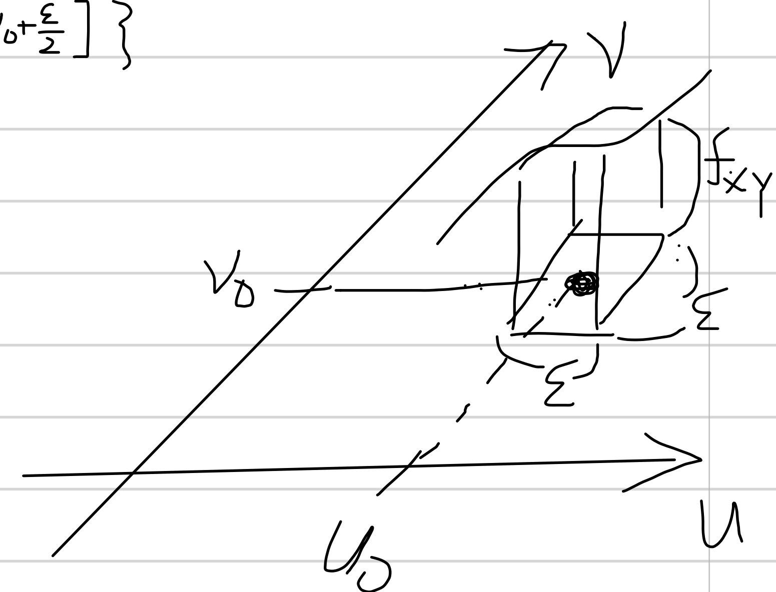
$\Rightarrow P\{(X,Y)\in[u_0-\frac{\varepsilon}{2}, u_0+\frac{\varepsilon}{2}]\times[v_0-\frac{\varepsilon}{2}, v_0+\frac{\varepsilon}{2}]\}$

$$=\int_{u_0-\frac{\varepsilon}{2}}^{u_0+\frac{\varepsilon}{2}}\int_{v_0-\frac{\varepsilon}{2}}^{v_0+\frac{\varepsilon}{2}} f_{XY}(u,v)\,dv\,du$$

$$\approx \varepsilon^2 f_{XY}(u_0,v_0)+o(\varepsilon^2)$$

↳ for small ε

= a "piece-of-cake" problem



Properties of joint pdf

$f_{XY}(u,v) \le 1$? Not necessarily!
why? f_{XY} is not a probability but a density!

1. $f_{XY}(u,v) \ge 0$ for all $u, v \in \mathbb{R}$
2. $\iint_{\mathbb{R}^2} f_{XY}(u,v)\,du\,dv = 1$ → Volume of entire cake must be 1.

Joint CDF ⟺ Joint pdf

$$F_{XY}(u_0,v_0) = \int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{XY}(u,v)\,dv\,du$$

take partial derivatives w.r.t. u_0 and v_0

$$\Leftrightarrow \frac{\partial F_{XY}(u,v)}{\partial u_0 \partial v_0} = f_{XY}(u_0,v_0)$$

Marginal pdf

$$F_X(u_0) = \lim_{v_0\to\infty} F_{XY}(u_0,v_0) = F_{X,Y}(u_0,\infty) = \int_{-\infty}^{u_0}\int_{-\infty}^{\infty} f_{XY}(u,v)\,du\,dv$$

⇒ Differentiate w.r.t. u_0

$$f_X(u_0) = \int_{-\infty}^{\infty} f_{XY}(u_0,v)\,dv$$

← Area of the cut of the cake along the line $u = u_0$.

Conditional pdf

$$f_{Y|X}(v|u_0) \triangleq \frac{f_{XY}(u_0,v)}{f_X(u_0)}$$

← height at (u_0, v)

← area of the cut along the line $u = u_0$.

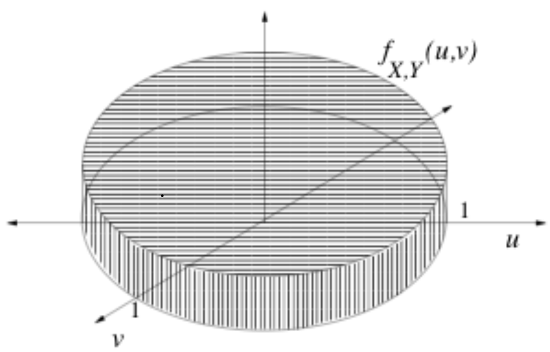
$$\Rightarrow \int_{-\infty}^{\infty} f_{Y|X}(v|u_0)\,dv = \frac{\int_{-\infty}^{\infty} f_{XY}(u_0,v)\,dv}{f_X(u_0)} = \frac{f_X(u_0)}{f(u)} = 0$$

Conditional pdf $f_{Y|X}(v|u_0)$ is well-defined only when $f_X(u_0) > 0$

⟺ only for u_0 in the support $f_X(u_0)$

The support of $f_X(u) = \{u \in \mathbb{R} \mid f_X(u) > 0\}$

Ex) $f_{XY} = \begin{cases} \frac{1}{\pi}, & u^2+v^2 \le 1 \\ 0 & \text{else} \end{cases}$



Q1) $P\{X \ge 0, Y \ge 0\}$? A: $\frac{1}{4}$

Q2) $P\{X^2+Y^2 \le r^2\}$

$$\frac{\text{Volume of the cake within the circle of radius } r}{\text{Volume of entire cake}} = \begin{cases} \frac{\pi r^2 \times \frac{1}{\pi}}{1} = r^2 & \text{if } r \in [0,1] \\ 0 & \text{if } r<0 \\ 1 & \text{if } r>1 \end{cases}$$

don't forget

Q3) pdf of X?

$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv = \begin{cases} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{1}{\pi}\,dv = \frac{2\sqrt{1-u_0^2}}{\pi} & \text{if } |u_0| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Q4) $f_{Y|X}(v|u_0)$?

i) What is the support of $f_X(u)$?

$[-1, 1] \rightarrow$ ~~No~~! $f_X(-1) = f_X(1) = 0$!

$(-1, 1) \rightarrow$ Yes

$$f_{Y|X}(v|u_0) = \begin{cases} \frac{f_{XY}(u_0,v)}{f_X(u_0)} = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-u_0^2}}{\pi}} = \frac{1}{2\sqrt{1-u_0^2}} & \text{if } -\sqrt{1-u_0^2} < v < \sqrt{1-u_0^2} \\ 0, & \text{else.} \end{cases}$$

only for $u_0 \in (-1, 1)$

EX) $f_{XY}(u,v) = \begin{cases} u + Av & 0 \le u \le 1, 0 \le v \le u \\ 0 & \text{else.} \end{cases}$

Q1) $f_X(u)$?

$$f_X(u) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv = \int_0^u u + Av\,dv = u^2 + A\frac{u^2}{2} = u^2\left(1+\frac{A}{2}\right) & \text{if } u \in [0,1] \\ 0, & \text{else} \end{cases}$$

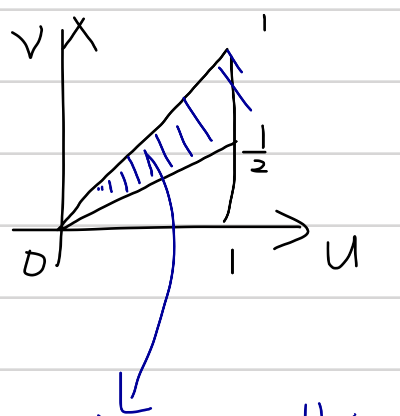
Q2) A? $\int_{-\infty}^{\infty} f_X(u)du = 1 \Rightarrow \int_0^1 u^2\left(1+\frac{A}{2}\right)du = 1 \Rightarrow \left(1+\frac{A}{2}\right)\frac{1}{3} = 1$

$\Rightarrow A = 4.$

Q3) $f_{Y|X}(v|u_0)$? $f_X(u_0) > 0$ if $0 < u \le 1$. Thus, for $0 < u \le 1$,

$$f_{Y|X}(v,u_0) = \begin{cases} \frac{u_0 + v}{3u_0^2} & \text{if } 0 \quad v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Q4) $P\{2Y \ge X\} = \int_0^1 \int_{\frac{u}{2}}^{u} u + 3v\,dv\,du = \int_0^1 u \cdot \frac{u}{2} + \frac{3}{2}\left(u^2 - \left(\frac{u}{2}\right)^2\right)du$



$$= \int_0^1 \frac{u^2}{2} + \frac{3}{2} \cdot \frac{3}{4} u^2 du = \int_0^1 \frac{13}{8} u^2 du$$

$$= \frac{13}{8} \frac{u^3}{3} \Big|_0^1 = \underline{\frac{13}{24}}$$

Lecture 28

4.4. Independence of Random Variables

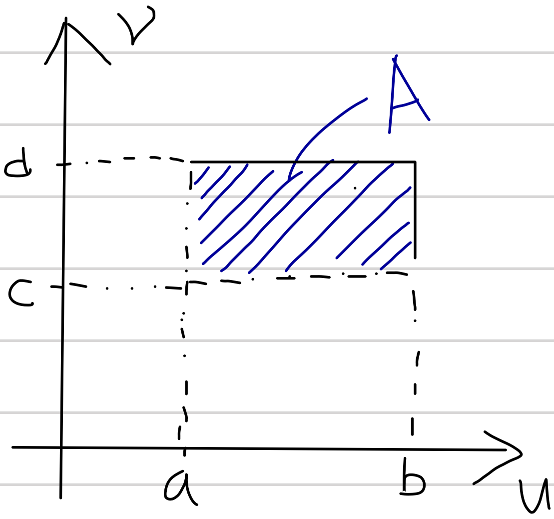
Def) X and Y are independent if $P\{X\in A, X\in B\} = P\{X\in A\}P\{Y\in B\}$ for any A and $B \in \mathbb{R}$.

$\Rightarrow$ Whether or not $X\in A$ is independent of whether or not $Y\in B$.

$$P\{X\le u, Y\le v\} = P\{X\le u\}P\{Y\le v\} \Rightarrow F_{XY}(u,v) = F_X(u)F_Y(v)$$

Converse is also true

If $F_{XY}(u,v) = F_X(u)F_Y(v)$



$$\begin{aligned}
\Rightarrow P[(X,Y)\in A] &= P[X\in(a,b], Y\in(c,d]] \\
&= F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c) \\
&= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) \\
&\quad + F_X(a)F_Y(c) \\
&= (F_X(b)-F_X(a))(F_Y(d)-F_Y(c)) \\
&= P\{a<X\le b\}P\{c<Y\le d\}
\end{aligned}$$

*Thus, X and Y are independent $\Leftrightarrow F_{X,Y}(u,v) = F_X(u)F_X(v)$ $\forall u,v\in\mathbb{R}$

Equivalently, for jointly continuous r.v.s X and Y,

X and Y are independent $\Leftrightarrow f_{XY}(u,v) = f_X(u)\cdot f_Y(v)$ $\quad\forall u,v\in\mathbb{R}$

For discrete type r.v.s X and Y

X and Y are independent $\Leftrightarrow p_{XY}(u,v) = p_X(u)p_Y(v)$ $\quad\forall u,v\in\mathbb{R}$

For given cdf Fxy, pdf fxy, or pmf pxy, how can we determine whether X and Yare independent?

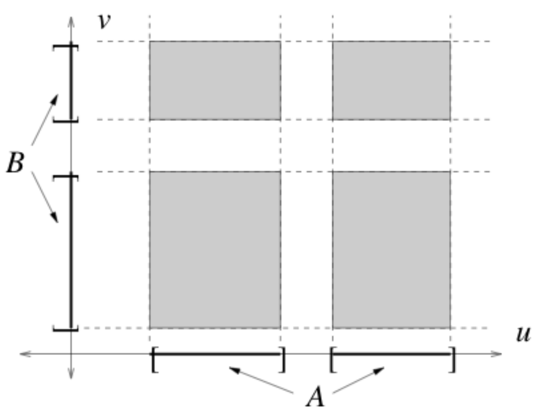
Proposition 4.4.4) If X and Y are independent, the support of fxy (or pxy) is a product set.

What is a product set?

A set S is called a product set if

$(a,b) \in S$ and $(c,d) \in S$

$\Rightarrow (a,d) \in S$ and $(b,c) \in S$



. Equivalently, if the support of fxy (u,v) is not a product set, X and Y are not independent.

"How to tell whether X and Y are independent?"

Q1) Is the support of X and Y a product set? $\xrightarrow{\text{No}}$ Dependent

↓ Yes!

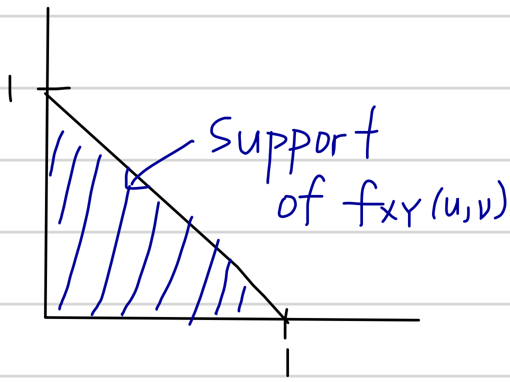
Does $f_{XY}(u,v)$ (or $p_{XY}(u,v)$) factors into a function of u and a function of v? $\xrightarrow{\text{No}}$ Dependent

e.g. $f_{XY}(u,v) = g_1(u)\, g_2(v)$

↓ Yes

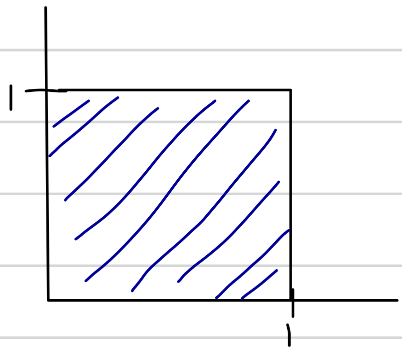
Independent

Ex) $f_{XY}(u,v) = \begin{cases} \frac{u^2 v^2}{c} & \text{if } u+v \le 1 \text{ and } u,v \ge 0 \\ 0 & \text{else} \end{cases}$



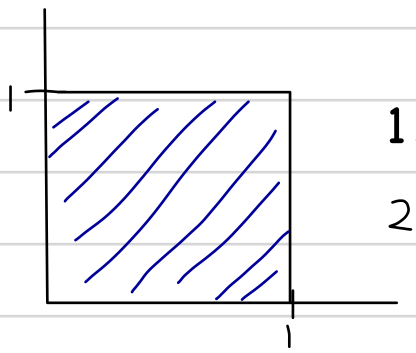
Since the support is not a product set, X and Y are dependent!

Ex) $f_{XY}(u,v) = \begin{cases} u+v & \text{if } u \in [0,1], v \in [0,1] \\ 0 & \text{else} \end{cases}$



1. The support is a product set.
2. fxy(u, v) does not factor! ⇒ X and Y are dependent!

Ex) $f_{XY}(u,v) = \begin{cases} 9u^2 v^2 & \text{if } u \in [0,1], v \in [0,1] \\ 0 & \text{else.} \end{cases}$



1. The support is a product set!
2. $f_{XY}(u,v) = 9u^2v^2 = \underbrace{(3u^2)}_{= f_X(u)} \underbrace{(3v^2)}_{= f_Y(v)}$

⇒ X and Y are independent.

Proposition 4.4.2) X and Y are independent if and only if
$f_X(u)=0$ or $f_{Y|X}(v|u)=f_Y(u)$ $\forall u,v \in \mathbb{R}$

proof) X and Y are independent $\Leftrightarrow f_{XY}(u,v)=f_X(u)f_Y(v)$ $\forall u,v$.
Need to show

$f_{XY}(u,v)=f_X(u)f_Y(v)$, $\forall u,v \Leftrightarrow f_X(u)=0$ or $f_{Y|X}(v|u)=f_Y(v)$ $\forall u,v$

($\Leftarrow$) If $f_X(u)>0$, then $f_{XY}(u,v)=f_X(u)f_{Y|X}(v|u)=f_X(u)f_Y(u)$.
If $f_X(u)=0$, suppose that $f_{XY}(u,v)>0$.

$$0=f_X(u)=\int_{-\infty}^{\infty} f_{XY}(u,v)dy$$

$\Rightarrow f_{XY}(u,v)$ is not continuous in v. $\Rightarrow$ Contradiction
hence, $f_{XY}(u,v)=0=f_X(u)f_Y(v)$

($\Rightarrow$) $f_X(u)>0$, $f_{Y|X}(v|u)$ exists

$$f_{Y|X}(v|u)=\frac{f_{XY}(u,v)}{f_X(u)}=\frac{f_X(u)f_Y(v)}{f_X(u)}=f_Y(u).$$

$f_X(u)=0$, $f_{XY}(u,v)=f_X(u)f_Y(v) \Rightarrow f_X(u)=0$ automatically!