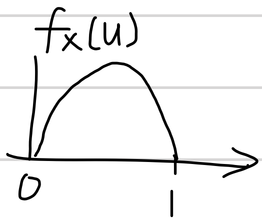
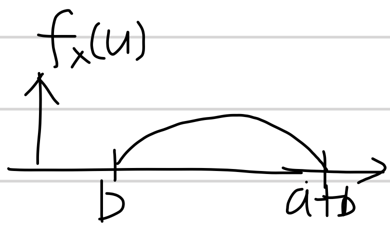
Lecture 23

Recall linear scaling

Suppose a r.v. X $\Rightarrow$ $Y = aX + b$



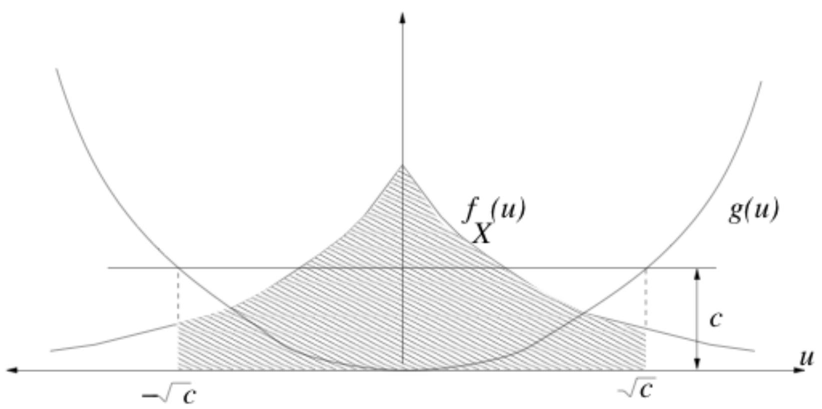
$\Longrightarrow$



In general, $f_Y(u) = f_X\left(\frac{u-b}{a}\right)\frac{1}{a}$.

What if $Y = g(X)$ where g is not linear.

Ex) $Y = X^2$, $f_X(u) = \frac{1}{2}e^{-|u|}$



How to find $f_Y(u)$? Consider CDF first!

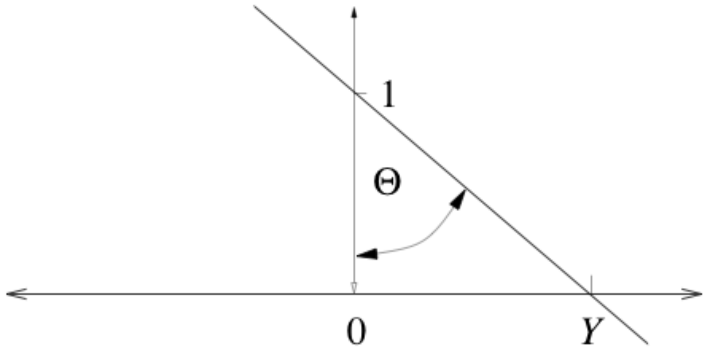
$$F_Y(u) = P(Y \le u) = P(X^2 \le u) = \begin{cases} 0 & \text{if } u < 0, \\ P(-\sqrt{u} \le X \le \sqrt{u}) & \text{if } u \ge 0 \end{cases}$$

$\Rightarrow$ For $u \ge 0$, $F_Y(u) = 1 - 2F_X(\sqrt{u}) \Rightarrow f_Y(u) = \cancel{2} \cdot f_X(\sqrt{u}) \cdot \frac{1}{\cancel{2}\sqrt{u}}$

$$\Rightarrow f_Y(u) = \begin{cases} \frac{1}{2\sqrt{u}} e^{-\sqrt{u}} & u \ge 0 \\ 0 & u < 0 \end{cases}$$

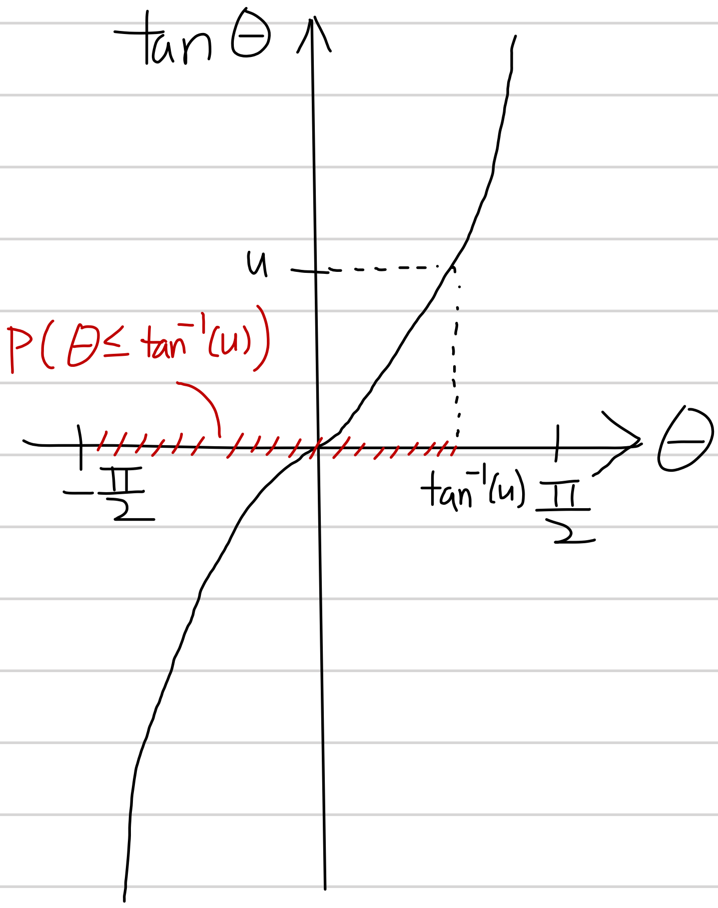
Exercise: $E[Y]$, $Var[Y]$?

Ex)



$\Theta \sim \text{uniform}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

Q1) Y? $\quad Y = \tan\Theta \quad$ (i.e, $g(u) = \tan(u)$)



Q2) $f_Y(u)$?

$F_Y(u) = P(Y \le u) = P(\tan\theta \le u)$

$= P(\theta \le \tan^{-1}(u))$

$= \dfrac{\frac{\pi}{2} + \tan^{-1}(u)}{\pi}$

$f_Y(u) = \dfrac{d}{du}\dfrac{\frac{\pi}{2} + \tan^{-1}(u)}{\pi}$

$= \dfrac{1}{\pi(1+u^2)}$ "Cauchy's distribution"

How to find $\tan^{-1}(u)'$

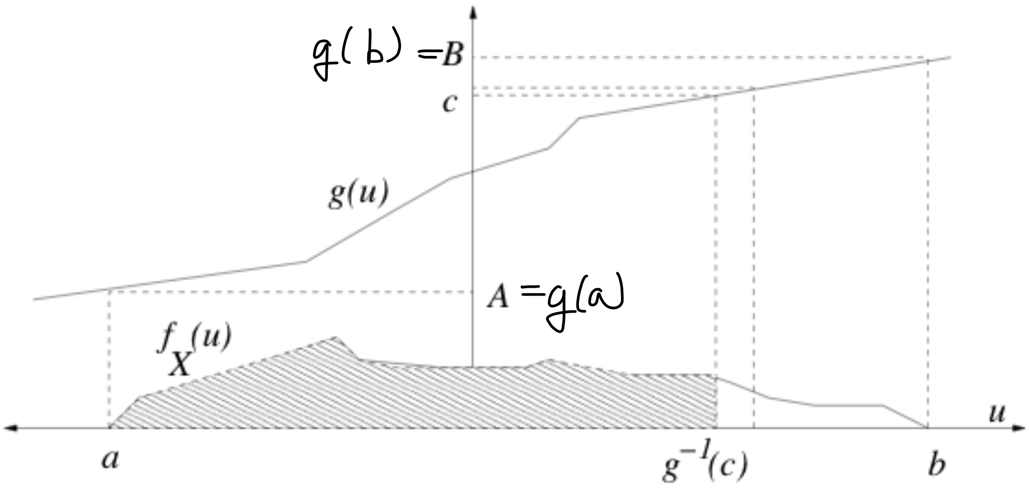
$y = \tan^{-1}(u) \Rightarrow \tan y = u. \Rightarrow \frac{d}{du}\tan y = 1 \Rightarrow \frac{dy}{du}\frac{d}{dy}\tan y = 1$

$\Rightarrow \frac{dy}{du}\sec y^2 = 1 \Rightarrow \frac{dy}{du}(1+u)^2 = 1 \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$

In general, $Y = g(X)$ where $X \in [a, b]$.

If g is differentiable and strictly monotone, on $[a, b]$

g' exists $\qquad$ g^{-1} exists



$$F_Y(u) = P(Y \le u) = P(g(X) \le u) = P(X \le g^{-1}(u))$$

$$= F_X(g^{-1}(u)) \qquad = \frac{1}{g'(g^{-1}(u))} \quad \text{why ?}$$

$$\Rightarrow f_Y(u) = f_X(g^{-1}(u)) \frac{d}{du} g^{-1}(u)$$

$$\Rightarrow f_Y(u) = \begin{cases} f_X(g^{-1}(u)) \dfrac{1}{g'(g^{-1}(u))} & \text{if } g(a) \le u \le g(b) . \\ 0 & \text{otherwise} \end{cases}$$

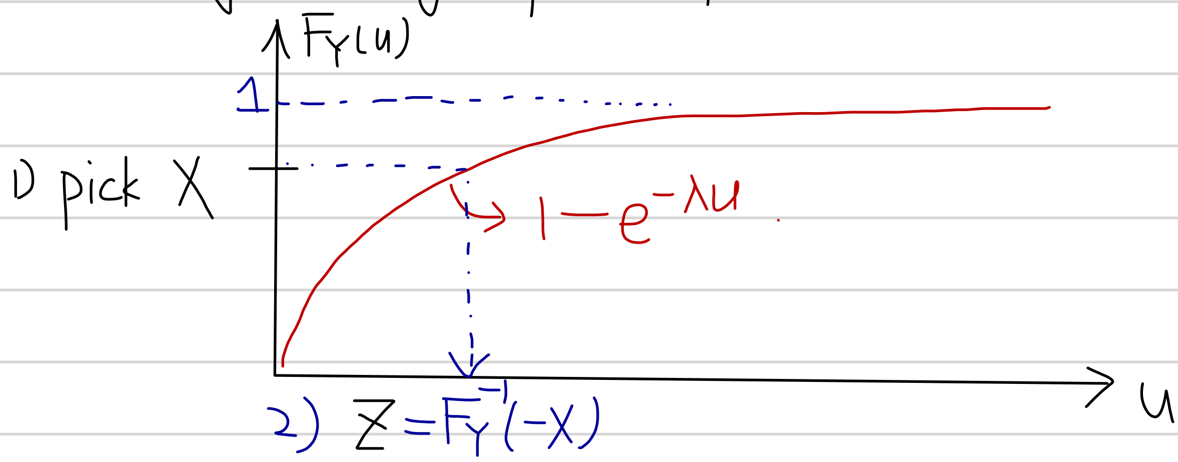
Ex) Random variable generator.

1) You want to generate a r.v. Y with monotonic CDF $F_Y(u)$.

2) You are given a random variable generator that generates $X \sim$ Uniform $[0, 1]$.

Q: How to generate Y using X? A: $Y = F_Y^{-1}(X)$.

ex) generating exp r.v. Y with λ.



Claim: Z and Y have the same distribution $\sim$ Uniform[0,1]

proof 1) $F_Z(u) = P(Z \le u) = P(F_Y^{-1}(X) \le u) = P(X \le F_Y(u))$

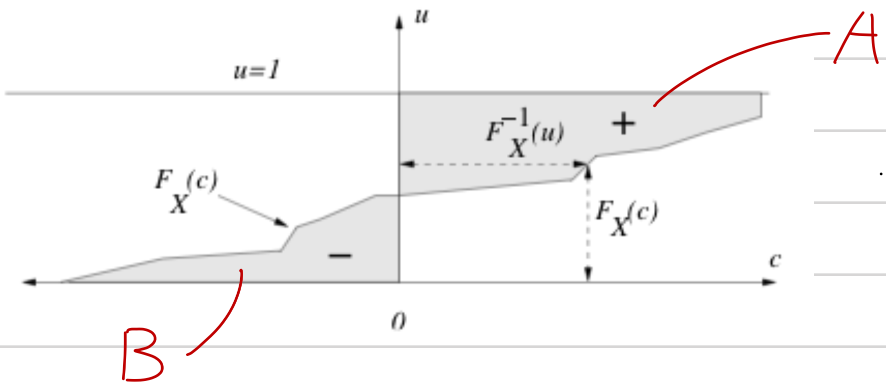
$= \dfrac{F_Y(u)}{1} = F_Y(u)$

proof 2) $f_Z(u) = f_X(g^{-1}(u))\,(g^{-1}(u))'$

$(g^{-1}(u))' = (F_Y(u))' = f_Y(u)$, $f_X(v) = 1$ for all $v \in [0,1]$

$\Rightarrow f_Z(u) = f_Y(u)$!

3.8.3 The area rule for expectation



$E[X] = A - B$! why?

proof: $A - B = \int_0^\infty (1 - F_X(c))dc - \int_{-\infty}^0 F_X(c)\,dc$

or

$\int_0^1 F_X^{-1}(u)\,du$.

From r.v. generator, $F_X^{-1}(U)$ where $U \sim \text{Uniform}[0,1]$ has the same distribution as X.

$$\Rightarrow E[X] = E[F_X^{-1}(U)] \underbrace{=}_{\text{LOTUS}} \int_0^1 \underbrace{f_U(u)}_{=1} \cdot F_X(u)\,du$$

$$= \int_0^1 F_X(u)\,du$$

3.9 Failure rate function.

T : Random variable denoting the lifetime of an item.

failure rate function

$$h(t) \triangleq \lim_{\varepsilon \to 0} \frac{P(t < T \le t+\varepsilon \mid T > \varepsilon)}{\varepsilon}$$

⇒ If the item is still working fine after t time unit
Probability that the item fails within the next ε time

$$\Rightarrow \quad h(t)\varepsilon + o(\varepsilon) \approx h(t)\varepsilon.$$

Another expression of $h(t)$

$$h(t) = \lim_{\varepsilon \to 0} \frac{P(t < T \le t+\varepsilon \mid T > t)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{P(t < T < t+\varepsilon, \cancel{t > T})}{P(t > T)\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{P(t < T < t+\varepsilon)}{P(t > T)\varepsilon} = \frac{f_T(t)}{1 - F_T(t)}$$

Solving a differential eq.

$$F_T(t) = 1 - e^{-\int_0^t h(s)ds}$$

EX) $T \sim \exp(\lambda)$, $h(t)$?

$$F_T(\lambda) = 1 - e^{-\lambda t}, \quad f_T(\lambda) = \lambda e^{-\lambda t} \Rightarrow h(t) = \lambda$$

The failure rate is a constant regardless of time.

⇒ Due to the memoryless property of exp. r.v.

Q2) If $h(t) = \frac{t}{\sigma^2}$ for $t \geq 0$, F_T and f_T ?

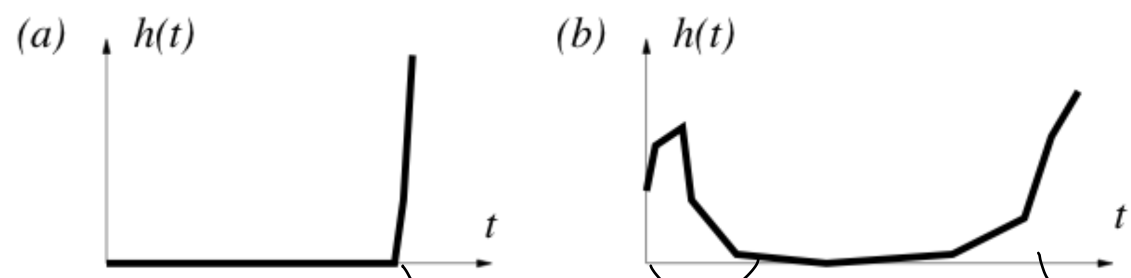
$$F_T(t) = 1 - e^{-\int_0^t \frac{s}{\sigma^2} ds} = 1 - e^{-\frac{t^2}{2\sigma^2}}$$

$$f_T(t) = e^{-t^2/2\sigma^2} \cdot \frac{2t}{2\sigma^2} = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}$$

$$\Rightarrow f_T(t) = \begin{cases} \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} & t \geq 0 \\ 0 & \text{o.w} \end{cases}$$

(Rayleigh)

Analysis of a failure rate function



zero rate   infinite rate   initial defect detection   aging

$\Rightarrow$ deterministic lifetime. $\Rightarrow$ Random lifetime

Determine the warranty of your company product to be th peak of the failure function!

3.10. Binary Hypothesis testing (Continuous type)

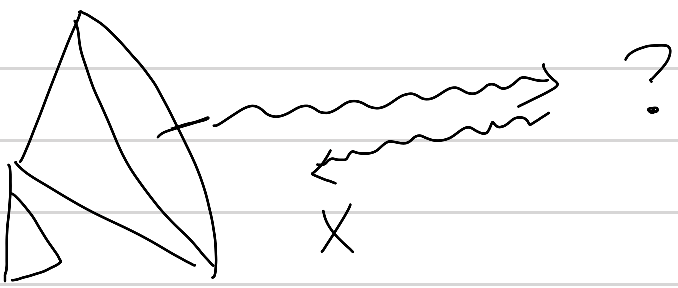
TWO HYPOTHESIS : H_0 & H_1

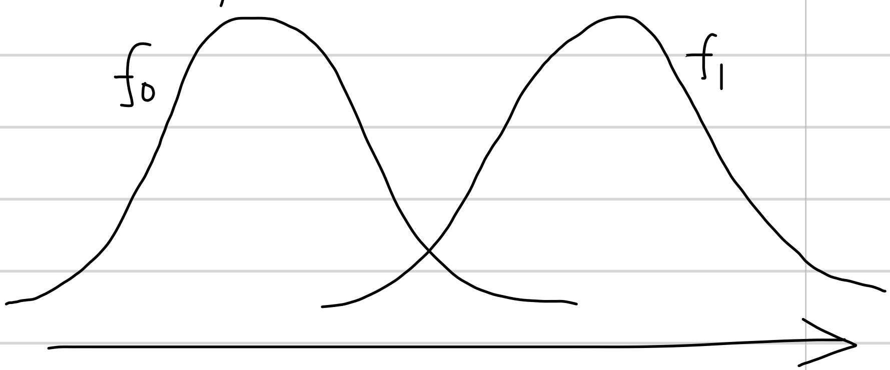
Outcome X is a continuous type r.v. only affected by either H_0 or H_1.

ex) Radar system to detect an enermy aircraft.

H_0 : No enemy aircraft

H_1 : Enemy aircraft present.





f_0 : pdf of X if H_0 is true

f_1 : pdf of X if H_1 is true

Observation : $X = j \Rightarrow$ Prob that $X \in [j - \frac{\varepsilon}{2}, j + \frac{\varepsilon}{2}] \approx \varepsilon f_i(j)$

Under Maximum likelihood decision

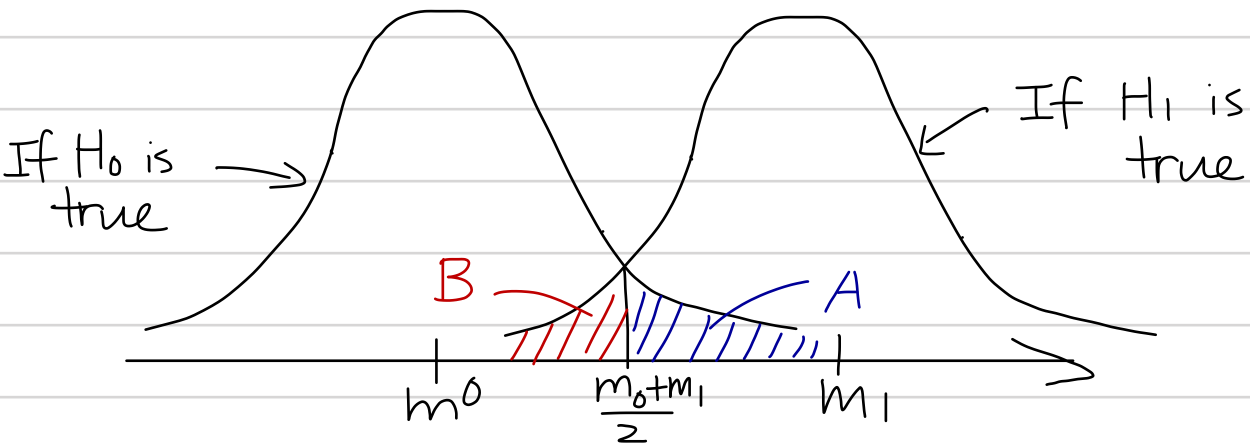
$f_0(j) > f_1(j) \Rightarrow$ Say H_0 is true

$f_1(j) < f_0(j) \Rightarrow$ Say H_1 is true

Likelihood ratio

$$\Lambda(j) = \frac{P[X \in [j - \frac{\varepsilon}{2}, j + \frac{\varepsilon}{2}] \mid H_1]}{P[X \in [j - \frac{\varepsilon}{2}, j + \frac{\varepsilon}{2}] \mid H_1]} = \frac{f_1(j)\cancel{\varepsilon}}{f_0(j)\cancel{\varepsilon}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Ex) If H_0 is true, $X \sim N(m_0, \sigma^2)$
If H_1 is true, $X \sim N(m_1, \sigma^2)$



ML: $H_0 \leftarrow | \rightarrow H_1$

ML Rule?

$$\Lambda(j) = \frac{f_1(j)}{f_0(j)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(j-m_1)^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(j-m_0)^2}{2\sigma^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left((j-m_1)^2-(j-m_0)^2\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Leftrightarrow \underbrace{-\frac{1}{2\sigma^2}}_{<0}(2j-m_1-m_0)(\underbrace{-m_1+m_0}_{<0}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln 1 = 0$$

$$\Leftrightarrow j \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m_1+m_0}{2}$$

$*$ $P_{\text{false alarm}}$ (or P_{error} conditioned on H_0)

$= P[\text{Say } H_1 \text{ is true} \mid H_0] = P\left[X > \frac{m_1+m_0}{2} \mid H_0\right] =$ "A" in the figure

(Since $X \sim N(m_0, \sigma)$) $= P\left[\frac{X-m_0}{\sigma} > \frac{m_1-m_0}{2\sigma} \mid H_0\right] = Q\left(\frac{m_1-m_0}{2\sigma}\right)$

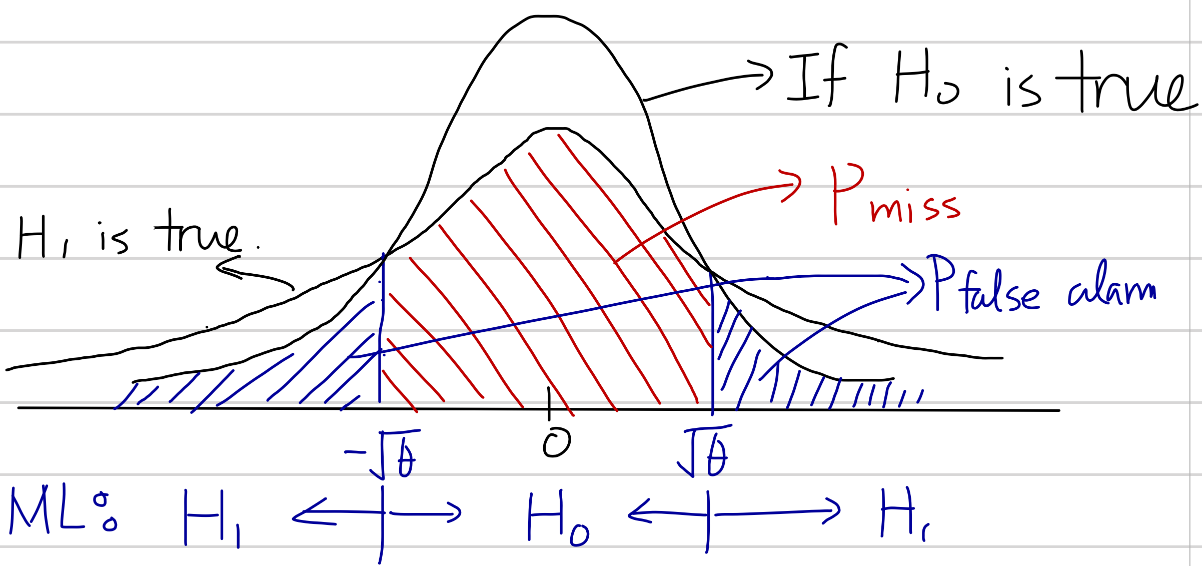
$*$ P_{miss} (or P_{error} condition on H_1)

$= P[\text{Say } H_0 \text{ is true} \mid H_1] = P\left[X < \frac{m_1+m_0}{2} \mid H_1\right] =$ "B" in the figure

(Since $X \sim N(m_1, \sigma)$) $= P\left[\frac{X-m_1}{\sigma} < \frac{m_0-m_1}{2\sigma} \mid H_0\right] = \Phi\left(\frac{m_0-m_1}{2\sigma}\right)$

$*$ P_{error}? Do not know. Why? No information about $P(H_0)$ & $P(H_1)$

EX) If H_0 is true, $X \sim N(0, \sigma_0)$
If H_1 is true, $X \sim N(0, \sigma_1)$ where $\sigma_1 > \sigma_0$



ML?

$$\Lambda(j) = \frac{f_1(j)}{f_0(j)} = \frac{\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{j^2}{2\sigma_1^2}\right)}{\frac{1}{\sqrt{2\pi}\sigma_0}\exp\left(-\frac{j^2}{2\sigma_0^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow \exp\left(\frac{j^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma_0}{\sigma_1}$$

$$\Rightarrow \frac{j^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\sigma_0}{\sigma_1}$$

$$\Rightarrow j^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\ln\frac{\sigma_0}{\sigma_1} \Big/ \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) := \theta$$

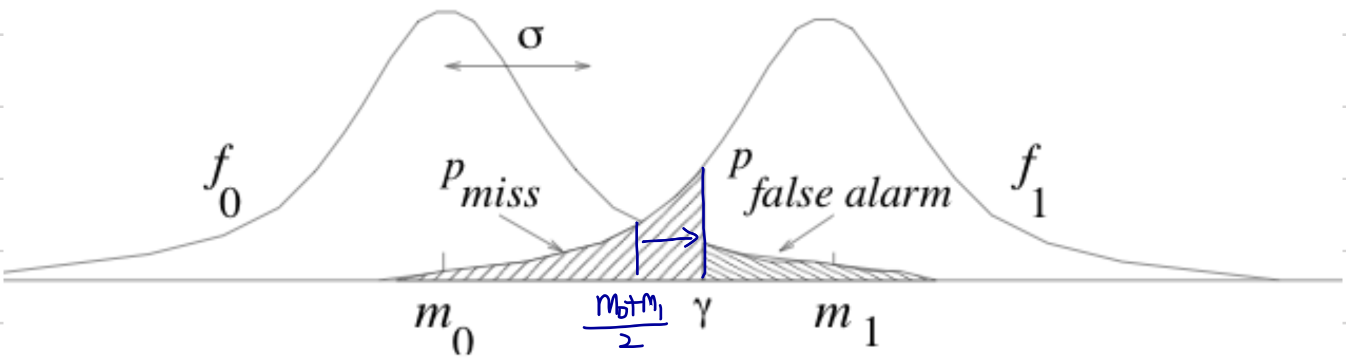
Hence, under ML, say H_0 is true if $j \in [-\sqrt{\theta}, \sqrt{\theta}]$
H_1 is true otherwise

$P_{false alarm} = P[\text{Say } H_1 \text{ is true} \mid H_0] = P[X \notin [-\sqrt{\theta}, \sqrt{\theta}] \mid H_0] = 2P[X > \sqrt{\theta}]$
(Since $X \sim N(0, \sigma_0)$) $= 2P\left(\frac{X}{\sigma_0} > \frac{\sqrt{\theta}}{\sigma_0}\right) = Q\left(\frac{\sqrt{\theta}}{\sigma_0}\right)$

$P_{miss} = P[\text{Say } H_0 \text{ is true} \mid H_1] = P[X \in [\sqrt{\theta}, \sqrt{\theta}] \mid H_1] = 1 - 2P[X > \sqrt{\theta}]$
(Since $X \sim N(0, \sigma_1)$) $= 1 - 2Q\left(\frac{\sqrt{\theta}}{\sigma_1}\right)$

P_{error}? Do not know

Can we test better with information of $\underbrace{P(H_0) \;\&\; P(H_1)}_{\text{aposteriori prob.}}$?



If $P(H_1) < P(H_0)$, i.e., H_1 happens rarely, the threshold must be moved in favor of H_0.

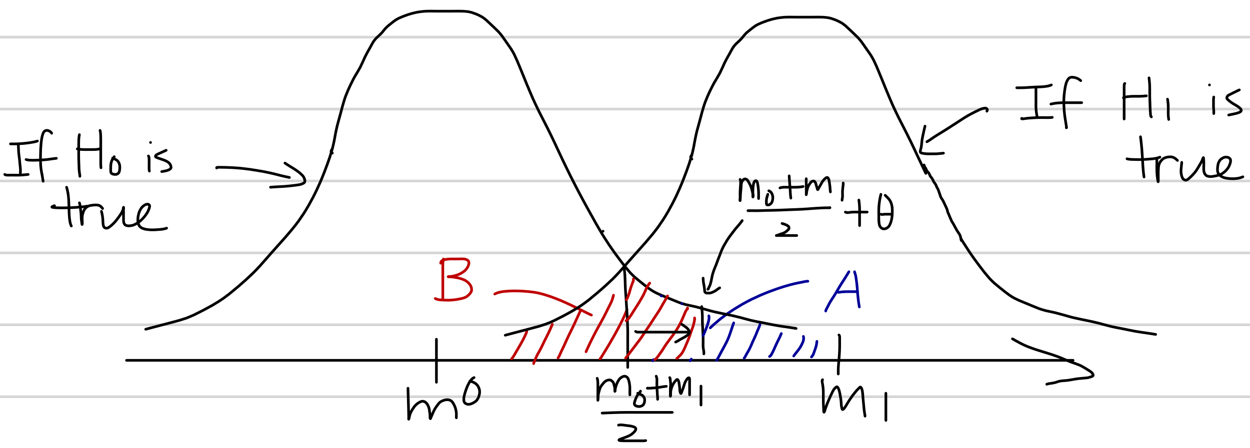
MAP rule for observed $X=j$

$$1 \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{P[X\in[j-\frac{\varepsilon}{2}, j+\frac{\varepsilon}{2}], H_1]}{P[X\in[j-\frac{\varepsilon}{2}, j+\frac{\varepsilon}{2}], H_0]} = \frac{P[X\in[j-\frac{\varepsilon}{2}, j+\frac{\varepsilon}{2}]\,|\,H_1]\,\overbrace{P(H_1)}^{=\pi_1}}{P[X\in[j-\frac{\varepsilon}{2}, j+\frac{\varepsilon}{2}]\,|\,H_0]\,\underbrace{P(H_0)}_{=\pi_0}}$$

$$\Rightarrow \frac{\pi_0}{\pi_1} \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{f_1(j)}{f_0(j)} = \Lambda(j)$$

Ex) If H_0 is true, $X \sim N(m_0, \sigma^2)$

If H_1 is true, $X \sim N(m_1, \sigma^2)$



ML: $H_0 \longleftarrow | \longrightarrow H_1$

MAP rule with $\pi_0 = \frac{2}{3}$ $\pi_1 = \frac{1}{3}$

$$\Lambda(j) = \frac{f_1(j)}{f_0(j)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(j-m_1)^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(j-m_0)^2}{2\sigma^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1} = 2$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left((j-m_1)^2-(j-m_0)^2\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} 2$$

$$\Longleftrightarrow \underbrace{-\frac{1}{2\sigma^2}}_{<0}(2j-m_1-m_0)\underbrace{(-m_1+m_0)}_{<0} \underset{H_0}{\overset{H_1}{\gtrless}} \ln 2$$

$$\Longleftrightarrow j \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m_1+m_0}{2} + \underbrace{\frac{2\sigma^2 \ln 2}{m_1-m_0}}_{\triangleq \theta}$$

Threshold is shifted by this amount

* $P_{\text{false alarm}}$ (or P_{error} conditioned on H_0)

$= P[\text{Say } H_1 \text{ is true} \mid H_0] = P\left[X > \frac{m_1+m_0}{2}+\theta \mid H_0\right] =$ "A" in the figure

(Since $X \sim N(m_0,\sigma)$) $= P\left[\frac{X-m_0}{\sigma} > \frac{m_1-m_0}{2\sigma} + \frac{\theta}{\sigma} \,\middle|\, H_0\right] = Q\left(\frac{m_1-m_0+2\theta}{2\sigma}\right)$

* P_{miss} (or P_{error} condition on H_1)

$= P[\text{Say } H_0 \text{ is true} \mid H_1] = P\left[X < \frac{m_1+m_0}{2}+\theta \mid H_1\right] =$ "B" in the figure

(Since $X \sim N(m_1,\sigma)$) $= P\left[\frac{X-m_1}{\sigma} < \frac{m_0-m_1+2\theta}{2\sigma} \,\middle|\, H_1\right] = \Phi\left(\frac{m_0-m_1+2\theta}{2\sigma}\right)$

* $P_{\text{error}} = \pi_0 \cdot P_{\text{false alarm}} + \pi_1 P_{\text{miss}} = \frac{2}{3} Q\left(\frac{m_1-m_0+2\theta}{2\sigma}\right) + \frac{1}{3}\Phi\left(\frac{m_0-m_1+2\theta}{2\sigma}\right)$