Lecture 18

Recall that the CDF of a r.v. X

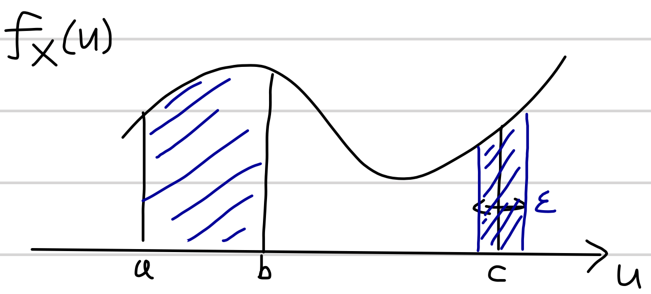
$$F_X(c) = P\{X \le c\}$$

Def) A r.v. X is a continuous type random variable if there is a function f_X such that

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$$

for all $c \in \mathbb{R}$. We call f_X *the probability density function*.

ex) for $a<b$, $P(a < X \le b) = \int_a^b f_X(u)\,du$



* $\int_{-\infty}^{\infty} f_X(u)\,du = 1$

* f_X is continuous in the domain

$\Rightarrow F_X$ is continuous & differentiable in the domain

* There exists a continuous r.v. with a discontinuous pdf, which is out of the scope of this course.

* $P(c - \frac{\varepsilon}{2} < X \le c + \frac{\varepsilon}{2}) = \int_{c-\varepsilon/2}^{c+\varepsilon/2} f_X(u)\,du \approx \int_{c-\varepsilon/2}^{c+\varepsilon/2} f_X(c)\,du$

$= \varepsilon \cdot f_X(c) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

* Mean of a continuous r.v. X.

$$\mu_X = E[X] = \int_{-\infty}^{\infty} u f_X(u)\,du.$$

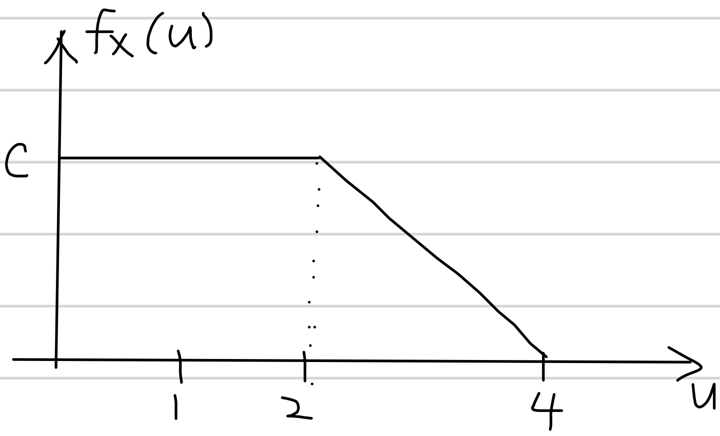
Lotus:

$$E[g(X)] = \int_{-\infty}^{\infty} g(u) f_X(u)\,du.$$

$$\sigma^2 = Var(X) = E[(X-\mu)^2]$$
$$= \int_{-\infty}^{\infty} (x-u)^2 f_X(x)\,dx = \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f_X(x)\,dx$$
$$= E[X^2] - 2\mu \int_{-\infty}^{\infty} x f_X(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f_X(x)\,dx$$
$$= E[X^2] - 2\mu\cdot\mu + \mu^2 = E[X^2] - \mu^2$$

* $Y = aX + b \Rightarrow E[Y] = aE[X] + b$, $Var(Y) = a^2 Var(X)$

Ex)

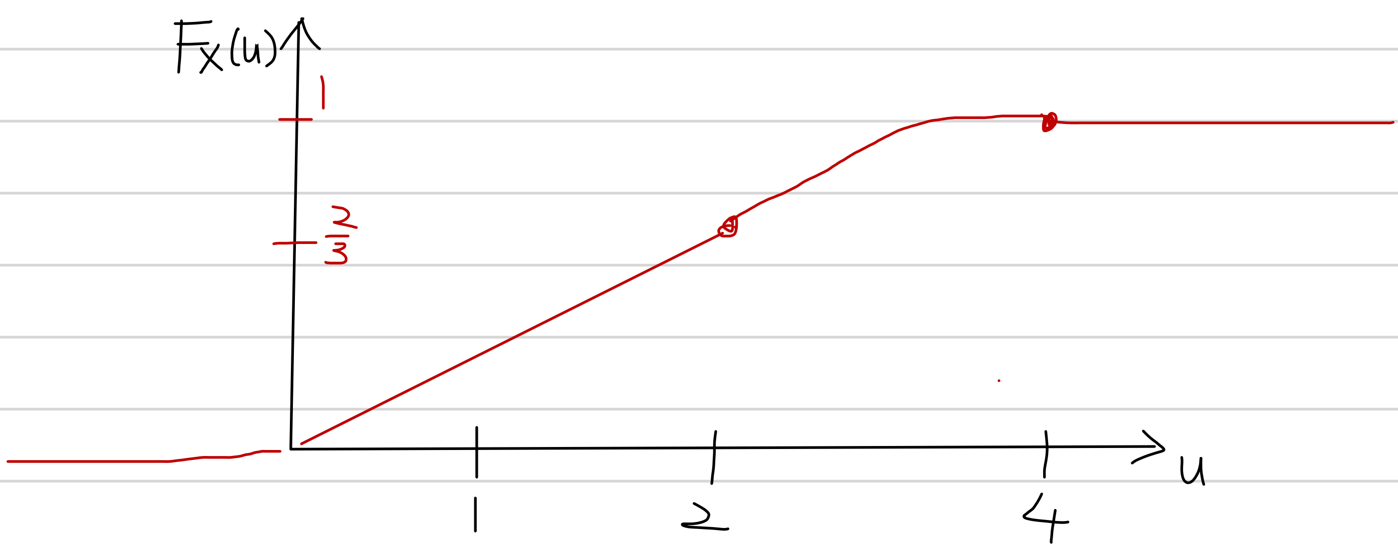


Q1) Find c
Q2) Find and sketch CDF
Q3) Find $P(X \geq 3 \mid X \geq 2)$
Q4) Find $Var(X)$

a) The area must be 1 $\Rightarrow 2c + c = 1 \Rightarrow c = \frac{1}{3}$

b)
$$F_X(u) = \begin{cases} 0 & \text{if } u < 0 \\ \frac{u}{3} & \text{if } 0 \leq u < 2 \\ ? & \text{if } 2 \leq u < 4 \\ 1 & \text{if } 4 \leq u \end{cases}$$

$$? = \int_{-\infty}^{u} f_X(x)\,dx = \frac{2}{3} + \int_2^u \left(2c - \frac{c}{2}x\right) dx = \frac{2}{3} + 2c(u-2) - \frac{c}{4}x^2 \Big|_2^u$$
$$= \frac{2}{3} + \frac{2}{3}(u-2) - \frac{1}{12}(u^2 - 4) = 1 + \frac{2}{3}(u-2) - \frac{u^2}{12} = \frac{-u^2 + 8u - 4}{12}$$



Q3) $P(X \geq 3 \mid X \geq 2) = \dfrac{P(X \geq 3, \cancel{X \geq 2})}{P(X \geq 2)} = \dfrac{1/12}{1/3} = \dfrac{1}{4}$

Q4) $E[X] = \int_0^4 u f_X(u)\,du = \int_0^2 u f_X(u)\,du + \int_2^4 u f_X(u)\,du$

$$= \int_0^2 \frac{1}{3} u\,du + \int_2^4 \left(\frac{2}{3}u - \frac{u^2}{6}\right) du$$

$$= \frac{1}{6}u^2 \Big|_0^2 + \frac{u^2}{3}\Big|_2^4 - \frac{u^3}{18}\Big|_2^4 = \frac{2}{3} + \frac{12}{3} - \frac{\cancel{56}^{\,28}}{\cancel{18}_{\,9}}$$

$$= \frac{42}{9} - \frac{28}{9} = \frac{14}{9}$$

Q5) $E[X^2] = \int_0^4 u^2 f_X(u)\,du = \int_0^2 \frac{u^2}{3}\,du + \int_2^4 u^2\left(\frac{2}{3} - \frac{u}{6}\right) du$

$$= \frac{u^3}{9}\Big|_0^2 + \frac{2}{9}u^3\Big|_2^4 - \frac{u^4}{24}\Big|_2^4$$

$$= \frac{8}{9} + \frac{2 \times 56}{9} - \frac{\cancel{16}^{\,2} \cdot 15}{\cancel{24}_{\,3}} = \frac{8 + 112 - 90}{9} = \frac{30}{9}$$

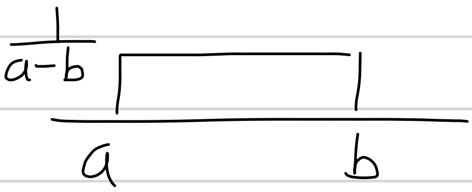
$$\mathrm{Var}(X) = \frac{10}{3} - \left(\frac{14}{9}\right)^2 \cong 0.913.$$

Lecture 19

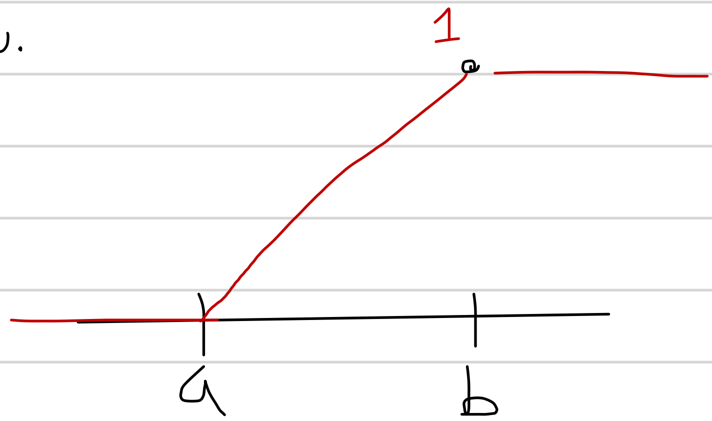
3.1 Uniform R.V.

Def) A random variable X is uniformly distributed in $[a, b]$ if

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{o.w.} \end{cases}$$



pdf



CDF

$$\ast\ E[X] = \int_{-\infty}^{\infty} u f_X(u)\,du = \int_a^b u \frac{1}{b-a}\,du = \frac{1}{b-a} \frac{u^2}{2}\Big|_a^b = \frac{b-a}{2}$$

$$E[X^2] = \int_a^b u^2 \frac{1}{b-a}\,du = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

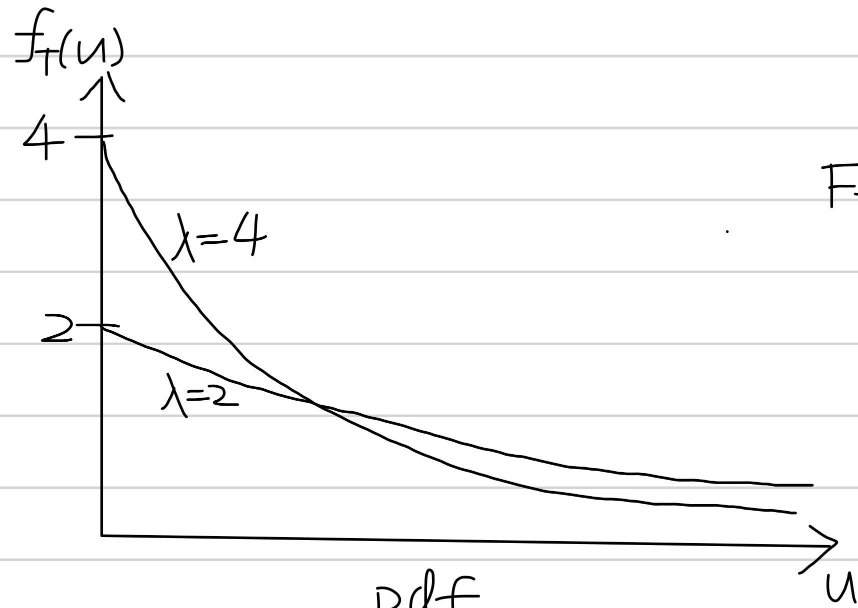
$$Var(X) = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \frac{b^2-2ab+a^2}{4} = \frac{1}{12}(b-a)^2$$

Exercise: For $k \ge 1$, $E[X^k]$?

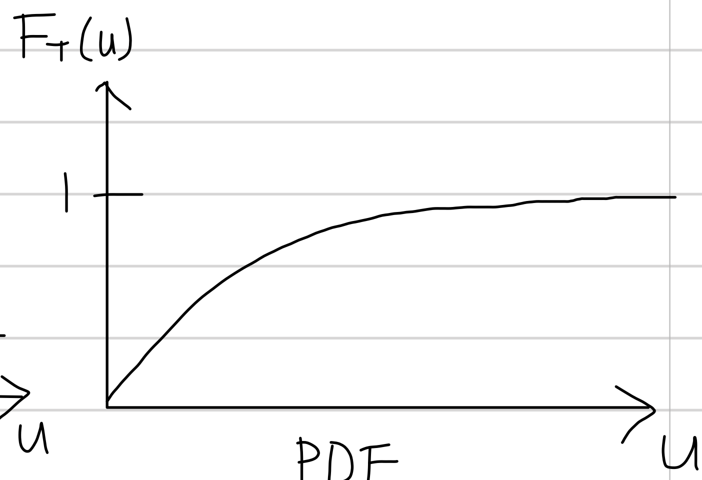
3.2 Exponential Distribution.

Def) A random variable T has the exponential distribution with parameter λ if its pdf is given by

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \ge 0 \\ 0, & \text{o.w.} \end{cases}$$



pdf



PDF

CDF: $F_T(t) = \begin{cases} 0 & \text{if } t<0 \\ \int_0^t \lambda e^{-\lambda s} ds = -e^{-\lambda s}\Big|_0^t = 1-e^{-\lambda t} \end{cases}$

$$P\{T \geq c\} = \int_c^\infty \lambda e^{-\lambda s} ds = -e^{-\lambda s}\Big|_t^\infty = e^{-\lambda t}$$

$$E[T^n] = \int_0^\infty t^n \lambda e^{-\lambda t} dt = \underbrace{-t^n e^{-\lambda t}\Big|_0^\infty}_{=0} + \int_0^\infty n t^{n-1} e^{-\lambda t} dt$$

$$= \frac{n}{\lambda} \int_0^\infty t^{n-1} \lambda e^{-\lambda t} dt = \frac{n}{\lambda} E[T^{n-1}] = \frac{n(n-1)}{\lambda^2} E[T^{n-2}] \cdots$$

$$\Rightarrow E[T^n] = \frac{n!}{\lambda^n}$$

Mean of T: $E[T] = \frac{1}{\lambda}$, $Var(T) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$$\Rightarrow \sigma = \frac{1}{\lambda}$$

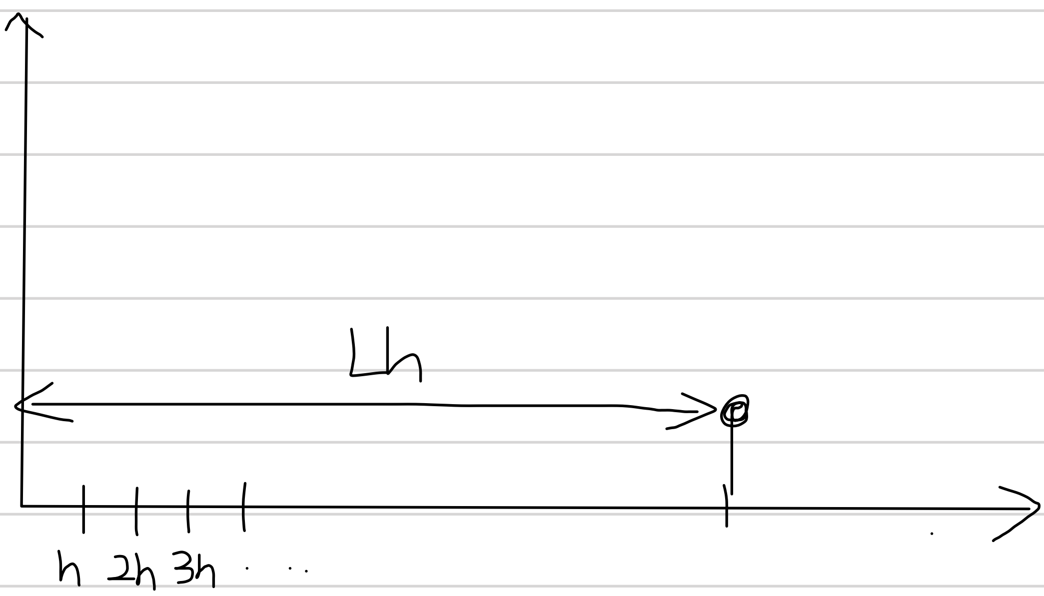
Memoryless Property

$$P[T>s+t \mid T>s] = \frac{P[T>s+t, T>s]}{P[T>s]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P[T>t]!$$

Two memoryless random variables

Geometric R.V. (Discrete)
Exponential R.V. (Continuous)) relationship?



$L \sim \text{Geometric}(p)$

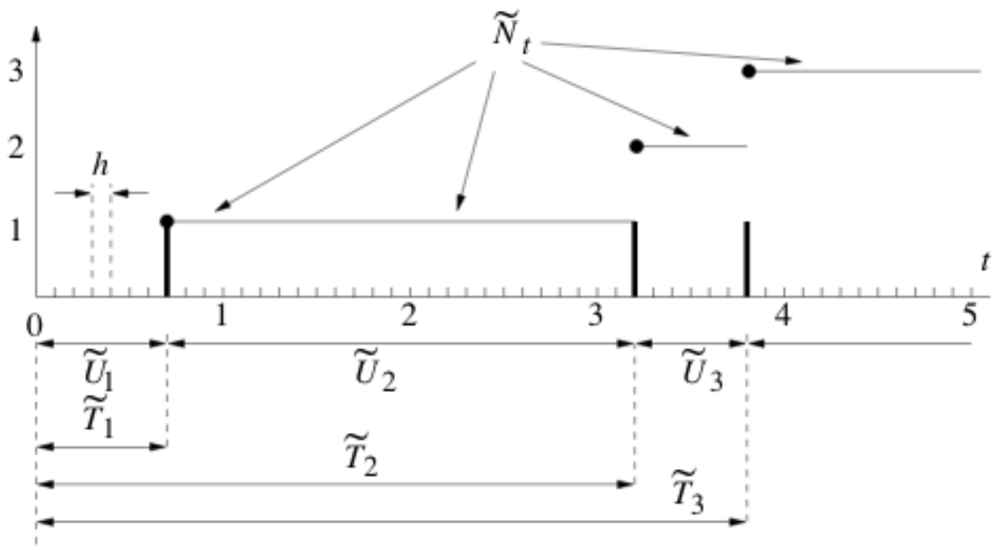
Time to get the first 1 in the Bernoulli process?

$T = L \cdot h$

$$P\{T > c\} = P\{L \cdot h > c\} = P\{L > \tfrac{c}{h}\} = (1-p)^{\lfloor \frac{c}{h} \rfloor}$$

$$(\text{If } p = \lambda h) \qquad = (1-\lambda h)^{\lfloor \frac{c}{h} \rfloor} \longrightarrow e^{-\lambda c} \quad (\text{Exp. R.V.})$$

Limit of scaled Bernoulli process.



$h \to 0$, $p \to 0$ s.t. $\lambda = \frac{p}{h}$.

X = # of trials to get the first success
$\sim$ Geometric(p)

T = time to get the first success
$\sim \text{Exp}(\lambda)$

Y = # of success by time t.
How many trials? $\lfloor \frac{t}{h} \rfloor$. Success prob $= \lambda h$.

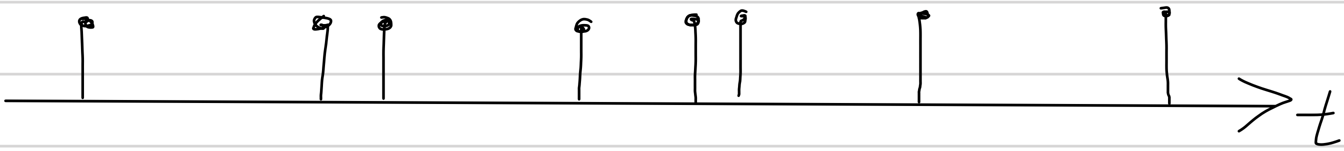
$$\Rightarrow Y \sim \text{Binomial}(\underbrace{\lfloor \tfrac{t}{h} \rfloor}_{=n}, \underbrace{\lambda h}_{=p})$$

As $h \to 0$, $n \to \infty$ and $p \to 0$ s.t. $np = \lambda t$.
Thus, $\text{Binomial}(\lfloor \frac{t}{h} \rfloor, \lambda h) \longrightarrow \text{Poisson}(\lambda t)$

Lecture 20

Poisson Process.

- Consider a random occurrence over time.



How can we characterize these random occurrences?

N_t : # of occurrences by time t.

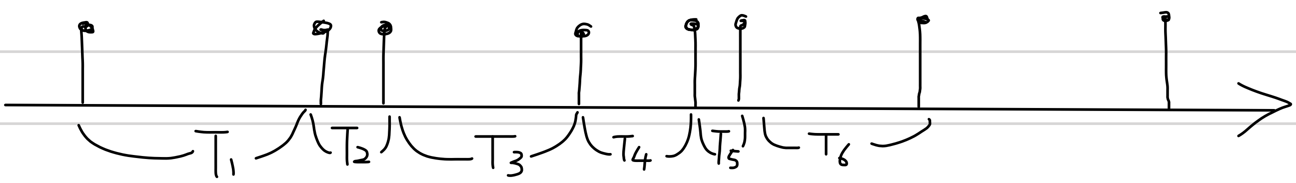
Def) N_t $(t \geq 0)$ is called a poisson process with rate λ if

(a) For any $[a,b]$ with $a \geq 0$, the number $N_b - N_a$ of occurrences in this interval satisfies

$$N_b - N_a \sim \text{Poisson}(\lambda(b-a))$$

(b) For any disjoint intervals $[a_1,b_1], [a_2,b_2], \cdots [a_k, b_k]$
$N_{b_1} - N_{a_1}, N_{b_2} - N_{a_2}, \cdots, N_{b_k} - N_{a_k}$ are all independent.

N_t is a poisson process with rate λ if and only if the intervals between occurrences are independent exponential random variables with parameter λ.



$T_1, T_2, \cdots$ are ind. exp. r.v.s with $\lambda \Leftrightarrow N_t$ is Poisson process with rate λ.

* For intervals $[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]$,
\# of occurrences in these intervals?

$\sim$ Poisson(λt_k)
where $t_k = \sum_{i=1}^{k} (b_i - a_i)$

Ex) Buses arrive at a bus stop according to a Poisson process at rate 6 per hour.

Q1) P[3 buses arrive in a min.]?

$\lambda = 0.1$ per min, time interval $= 1$ min.
X = # of buses in a minute. $\sim$ Poisson(0.1)

$$P[X=3] = \frac{0.1^3 e^{-0.1}}{3!} = 0.00015$$

Q2) P[At least one bus for each min for one hour]

$$= P(X \geq 1)^{60} \quad (\because \text{Bus arrival in each minute is independent!})$$
$$= (1 - P(X=0))^{60} = (1 - e^{-0.1})^{60}$$

Q3) P[At least 10 buses in an hour]

$Y \triangleq$ # of buses in one hour. $\Rightarrow E[Y] = 6$, $Var(Y) = 6$.
$\sim$ Poisson(6)

$$P[Y \geq 10] = \sum_{k=10}^{\infty} \frac{6^k e^{-6}}{k!} = 0.084$$

Markov: $P(Y \geq 10) \leq \frac{E[Y]}{10} = \frac{6}{10}$
Chebychev: $P(Y \geq 10) < P(|Y-6| > 4) < \frac{6}{16}$

Q4) P[Five buses in first hour | 10 buses in two hours]

X_1: # of buses in the first hour, X_2: # of buses in the second hour

$$P(X_1=5 \mid X_1+X_2=10) = \frac{P(X_1=5,\ X_1+X_2=10)}{P(X_1+X_2=10)}$$

$$P(X_1=5, X_2=5)$$
$$= P(X_1=5)\,P(X_2=5)$$) X_1 & X_2 are independent

$$= \frac{P(X_1=5)\,P(X_2=5)}{P(\underbrace{X_1+X_2}_{\text{Poisson}(12)}=10)} = \frac{\left(\frac{e^{-6}6^5}{5!}\right)^2}{\frac{e^{-12}12^{10}}{10!}} = \frac{\frac{6^5\,6^5}{5!\,5!}}{\frac{12^{10}}{10!}} = \left(\frac{1}{2}\right)^{10}\left(\frac{10!}{5!\,5!}\right)$$

$$= \left(\frac{1}{2}\right)^{10}\binom{10}{5}$$