Lecture 15

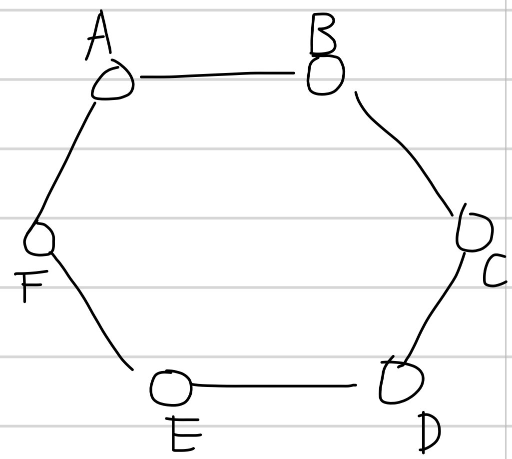
2.12 Reliability

Union bound : For any events $A_1, A_2, \cdots, A_m \in \mathcal{F}$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_m) \le P(A_1) + P(A_2) + \cdots + P(A_m)$$

Ex) Ring network.

i) n nodes & n links
ii) each link fails with prob. p.
iii) We call node i and node j disconnected if there is no path between both nodes.



Q1) Prob. that there is a pair of nodes are disconnected?

X = # of links failed $\sim$ Binomial(n, p)

$$\{X \ge 2\} = \{\text{there is a pair of peers disconnected}\}$$

$$\Rightarrow P\{X \ge 2\} = 1 - P\{X=0\} - P\{X=1\} = 1 - (1-p)^n - np(1-p)^{n-1}$$
$$= 1 - (1-p)^{n-1}(1-p+np)$$
$$= 1 - (1-p)^{n-1}(1+(n-1)p)$$

Q2) Union bound?

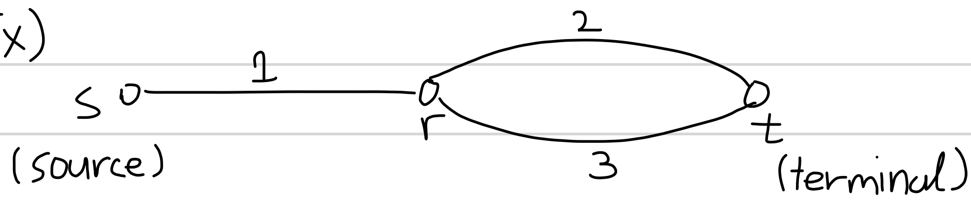
$$P\{X \ge 2\} = P\Big(\bigcup_{i=1}^{n} \bigcup_{j>i}^{n} A_{i,j}\Big) \quad \text{where } A_{i,j} = \{\text{links } i \text{ and } j \text{ fail}\}$$

$$= \binom{n}{2} P(A_{i,j}) = \binom{n}{2} p^2$$

If $n=6$, $p=0.001 \Rightarrow P\{X \ge 2\} = 1 - (0.999)^5(1.005) \approx 0.0000149\cdots$

Union bound $P\{X \ge 2\} \ge \frac{6\,5}{2}(0.001)^2 = 0.000015$

Ex)



i) Each link fails w.p. p. independently.

$F_i = \{\text{link } i \text{ fails}\}$

If there is no path of sucessful links from s to t,
We say that the network outage occurs.

$F = \{\text{network outage occurs}\}$

Q1) Express F with F_1, F_2, F_3

The network outage occurs if
link 1 fails, or
both link 1 and link 2 fail.

$\Rightarrow F = F_1 \cup F_2F_3$

Q2) P(F)?

$$P(F) = P(F_1 \cup F_2F_3) = P(F_1) + P(F_2F_3) - P(F_1F_2F_3)$$
$$= p + p^2 - p^3$$

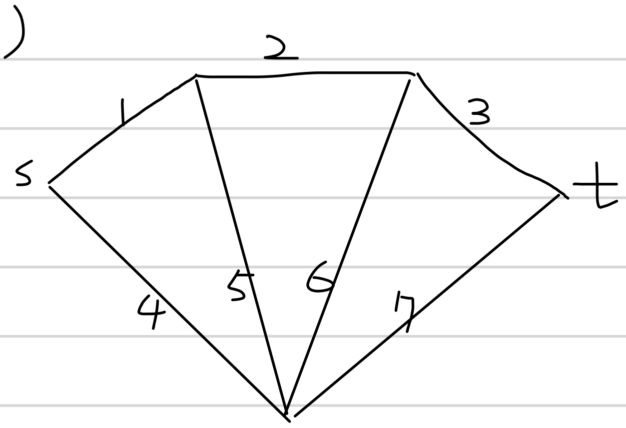
Q3) Union bound?

$$P(F) = P(F_1 \cup F_2F_3) \le P(F_1) + P(F_2F_3) = p + p^2$$

Q4) $$P(F_1|F) = \frac{P(F_1F)}{P(F)} = \frac{P(F_1 \cap (F_1 \cup F_2F_3))}{P(F)} = \frac{P(F_1 \cup F_1F_2F_3)}{P(F)}$$
$$= P(F_1)/P(F) = P/_{p+p^2-p^3} = \frac{1}{1+p-p^2}$$

Q5) $$P(F_3|F) = \frac{P(F_3 \cap F)}{P(F)} = \frac{P(F_3 \cap (F_1 \cup F_2F_3))}{P(F)} = \frac{P((F_3 \cap F_1) \cup F_2F_3)}{P(F)}$$
$$\frac{P(F_3 \cap (F_1 \cup F_2))}{P(F)} = \frac{p(2p-p^2)}{p+p^2-p^3} = \frac{2p-p^2}{1+p-p^2}.$$

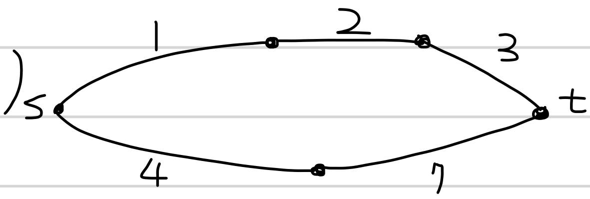
Ex)



i) Each link i fails w.p. p_i
$(q_i = 1 - p_i)$
ii) $F_i = \{\text{link } i \text{ fails}\}$
iii) $F = \{\text{network outage occurs}\}$

i) Condition on $F_5 \cap F_6$ (i.e., links 5 and 6 fail)

$$P(F \mid F_5 F_6) = P((F_1 \cup F_2 \cup F_3) \cap (F_4 \cup F_7) \mid F_5 F_6)$$



F_5, F_6 and F_1, F_2, F_3, F_4, F_7 are indep.

$$= P((F_1 \cup F_2 \cup F_3) \cap (F_4 \cup F_7))$$

(F_1, F_2, F_3) and (F_4, F_7) are independent!

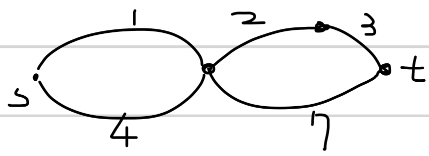
$$= P(F_1 \cup F_2 \cup F_3) \cdot P(F_4 \cup F_7)$$

de Morgan's law

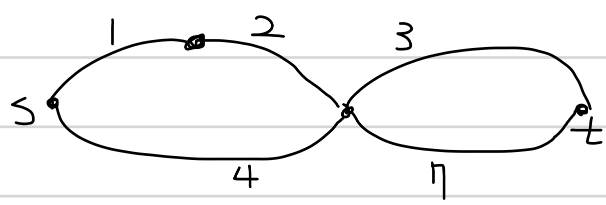
$$= (1 - P(F_1^c \cap F_2^c \cap F_3^c)) \cdot (1 - P(F_4^c \cap F_7^c))$$
$$= (1 - q_1 q_2 q_3)(1 - q_4 q_7)$$

ii) Conditioned on $F_5^c \cap F_6$ (i.e., link 6 fails and link 5 succeeds)

$$P(F \mid F_5^c F_6) = P(F_1 F_4 \cup (F_2 \cup F_3) F_7)$$
$$= p_1 p_4 + (p_2 + p_3 - p_2 p_3) p_7$$
$$- p_1 p_4 (p_2 + p_3 - p_2 p_3) p_7$$

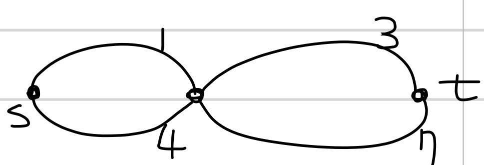


iii) Conditioned on $F_5 \cap F_6^c$



$$P(F \mid F_5 F_6^c) = (p_1 + p_2 - p_1 p_2) p_4$$
$$+ p_3 p_7 - p_3 p_7 (p_1 + p_2 - p_1 p_2) p_4$$

iv) Conditioned on $F_5^c \cap F_6^c$



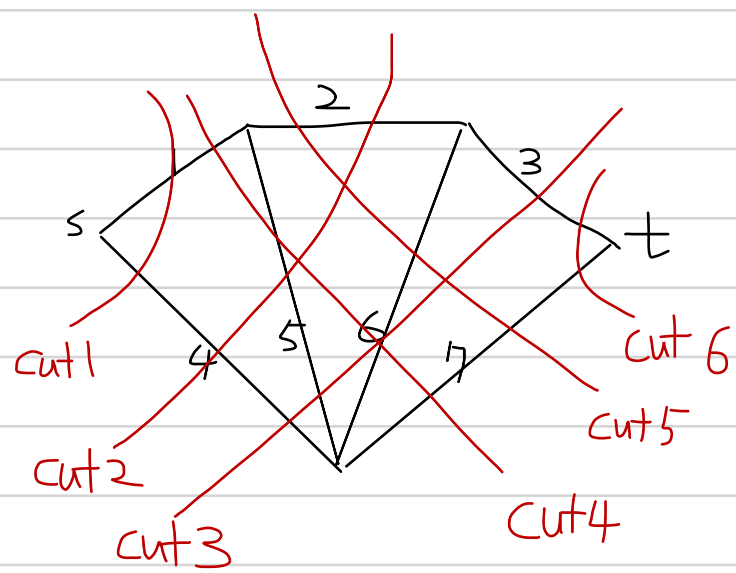
$$P(F \mid F_5^c F_6^c) = p_1 p_4 + p_3 p_7 - p_1 p_3 p_4 p_7$$

Overall

$$P(F) = P(F \mid F_5 F_6)\,P(F_5 F_6) + P(F \mid F_5^c F_6)\,P(F_5^c F_6)$$
$$\quad + P(F \mid F_5 F_6^c)\,P(F_5 F_6^c) + P(F \mid F_5^c F_6^c)\,P(F_5^c F_6^c)$$

$= ?$ exercise!

Analysis using Union bound



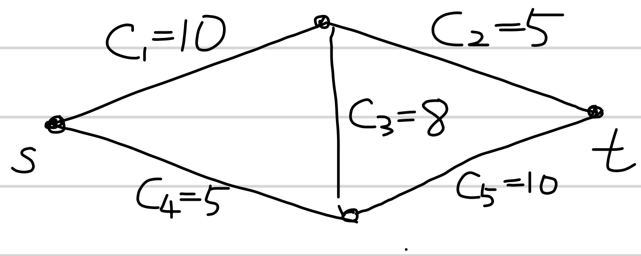
$\Rightarrow$ If all the links crossing cut i fail, there is no path from s to t, and thus the network is disconnected.

$E_i = \{$all the edges crossing cut i fail$\}$

$$P\{F\} \le P\{E_1 \cup E_2 \cup \cdots \cup E_6\} \le P(E_1) + P(E_2) + \cdots + P(E_6)$$
$$= p_1 p_4 + p_2 p_4 p_5 + p_3 p_4 p_5 p_6 + p_1 p_5 p_6 p_7 + p_2 p_6 p_7 + p_3 p_7$$

Lecture 15

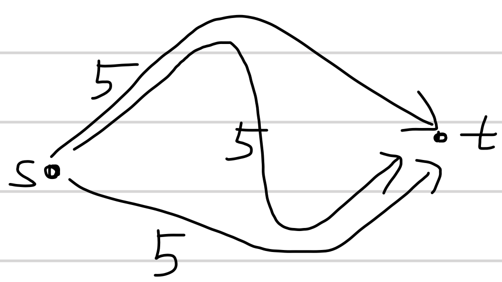
Ex) s-t flow network



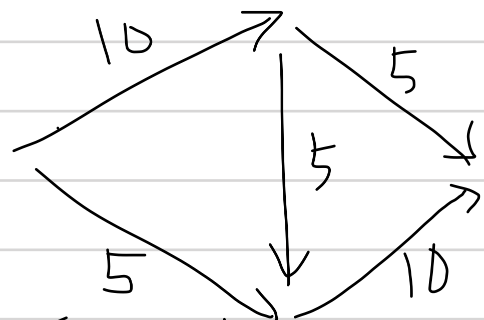
Consider a water pipe i that can flow C_i tons of water per second.

Q1) What is the max flow rate from s to t.

A) 15



path flow



Capacity used

Suppose that each pipe i fails (clogged) w.p. p_i $\quad (q_i = 1-p_i)$

X = max flow. Find the pmf of X.

$P\{X=15\} = P\{\text{no pipe fails}\} = q_1 q_2 q_3 q_4 q_5$

$P\{X=0\} = p_3((p_1+p_2-p_1p_2)(p_4+p_5-p_{45}) + q_3(p_1p_4+p_2p_5-p_1p_2p_4p_5)$

$P\{X=8\} = q_1 q_3 q_5 p_2 p_4$

$P\{X=10\} = q_1 q_5 \{p_2 q_3 q_5 + q_2 p_3 q_4 + q_2 q_3 p_4\}$

$P\{X=5\} = p_1 p_5 q_2 q_3 q_4 + q_1 p_3 q_2 + p_1 q_5 q_4 + q_1 q_5 p_8 (p_2 q_4 + q_2 p_4)$

How can we find all possible values of X?

State	capacity	probability
00000	30	$q_1q_2q_3q_4q_5$
00001	10	$q_1q_2q_3q_4p_5$
00010	20	$q_1q_2q_3p_4q_5$
00011	10	$q_1q_2q_3q_4p_5$
00100	10	$q_1q_2p_3q_4q_5$
00101	10	$q_1q_2p_3q_4p_5$
00110	10	$q_1q_2p_3p_4q_5$
00111	10	$q_1q_2p_3p_4p_5$
$\vdots$	$\vdots$	$\vdots$
11111	0	$p_1p_2p_3p_4p_5$

Construct the above table!

Ex) Analysis of an array code

Digital Communication

Sender $\xrightarrow[\text{49 bit message}]{\text{error}}$ Receiver — How can the receiver detect an error?

+ 15 bits for error dectection

original 49 bit message

0	1	1	1	1	0	1	1
1	0	0	0	1	0	0	0
1	0	1	0	0	1	0	1
0	1	0	0	1	1	0	1
1	0	1	1	1	0	1	1
0	1	1	1	1	1	0	1
0	1	1	1	0	0	1	0
1	0	1	0	1	1	1	1

additional 15-bit error-detection code.

Add 13 bits such that the # of ones in each column and each row is even.

Each bit flips w.p. 0.001.

ex) one error pattern

0	1	1	1	1	0	1	1
1	0	0	0	1	0	0	0
1	~~0~~ 1	1	0	0	1	0	1
0	1	0	0	1	~~1~~ 0	0	1
1	0	1	1	1	0	1	1
0	~~1~~ 0	1	1	1	1	0	1
0	1	1	1	0	0	1	0
1	0	1	0	1	1	1	1

How many different error patterns?

2^{64} (including the pattern with no error)

Q2) What is a necessary condition for an error pattern to be undetectable?

Y = # of errors

0	1	1	1	1	0	1	1	
1	0	0	0	1	0	0	0	
~~1~~ 0	0	1	0	0	1	0	1	→ odd 1's
0	1	0	0	1	1	0	1	
1	0	1	1	1	0	1	1	
0	1	1	1	1	1	0	1	
0	1	1	1	0	0	1	0	
1	0	1	0	1	1	1	1	
odd 1's								

Y=1 ⇒ detectable.

0	1	1	1	1	0	1	1	
1	0	0	0	1	0	0	0	
1	0	~~1~~ 0	0	~~0~~ 1	1	0	1	← even 1's
0	1	0	0	1	1	0	1	
1	0	1	1	1	0	1	1	
0	1	1	1	1	1	0	1	
0	1	1	1	0	0	1	0	
1	0	1	0	1	1	1	1	
		odd 1's		odd 1's				

Y=2 ⇒ detectable

	0	1	1	1	1	0	1	1
even 1's →	1	~~0~~ 1	0	0	~~1~~ 0	0	0	0
	1	0	1	~~0~~ 1	0	1	0	1
	0	1	0	0	1	1	0	1
	1	0	1	1	1	0	1	1
	0	1	1	1	1	1	0	1
	0	1	1	1	0	0	1	0
	1	0	1	0	1	1	1	1
		odd 1's		odd 1's	odd 1's			

Y=3 ⇒ detectable

0	1	1	1	1	0	1	1	
1	~~0~~ 1	0	0	~~1~~ 0	0	0	0	→ even
1	0	1	0	0	1	0	1	
0	~~1~~ 0	0	0	~~1~~ 0	1	0	1	→ even
1	0	1	1	1	0	1	1	
0	1	1	1	1	1	0	1	
0	1	1	1	0	0	1	0	
1	0	1	0	1	1	1	1	
	even			even				

Y=4 ⇒ undetectable

i) A detectable error pattern has at least four error bits

ii) More precisely, detectable error patterns have at least four error bits in the intersections of two rows and two columns.

Union bound from (i)

$P(\text{error detectable}) \le P\{Y \ge 4\} \le \binom{64}{4} p^4 = 0.953 \times 10^{-6}$

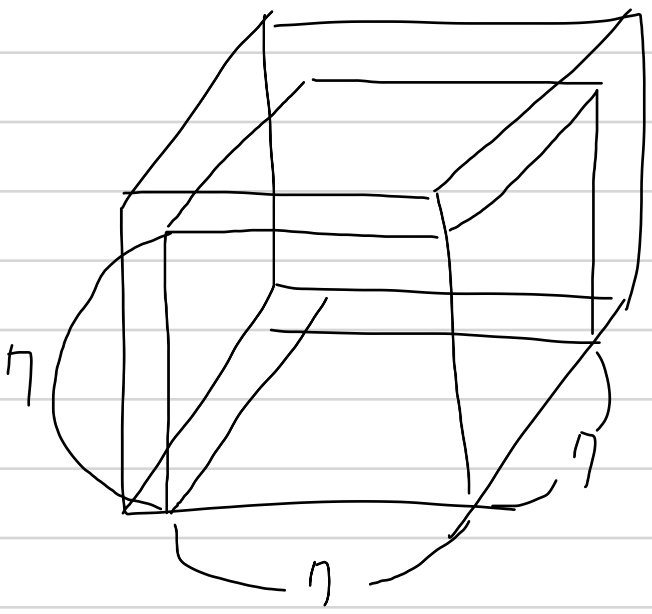
Union bound from (ii)

$P(\text{error detectable} \le P\{\text{there are four errors in the intersections of two rows and two columns}\}$

$\le \binom{\text{\# of ways to choose}}{\text{two rows and two columns}} \times p^4$

$= \binom{8}{2}\binom{8}{2} p^4 = 0.859 \times 10^{-9}$

Exercise: 3D-error-detection code



η^3-bit message.
$(8^3 - \eta^3)$-bit error detection bits.

Lecture 17

Chapter 3: Continuous-type random variables

- Classification of random variables based on the # of possible values of X

Finite	Countably infinite	uncountably infinite
Bernoulli Binomial	Geometric Negative Binomial Poisson	ex) $X \sim$ uniform $[0,1]$ $p_X(0.5)=0$, $p_X(0.3)=0$

How to express the distribution of X?

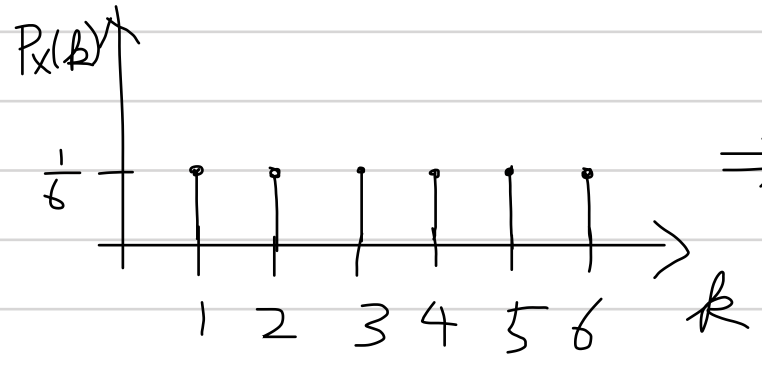
3.1 Cumulative distribution functions

Def) For a random variable X on $(\Omega, \mathcal{F}, P)$,

the cumulative distribution function of X is defined as

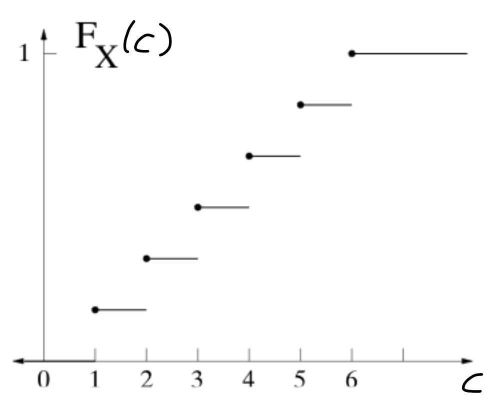
$$F_X(c) = P\{\omega : X(\omega) \le c\} = P\{X \le c\} \quad \text{(for short)}$$

Ex) X = # shown in a die.



pmf

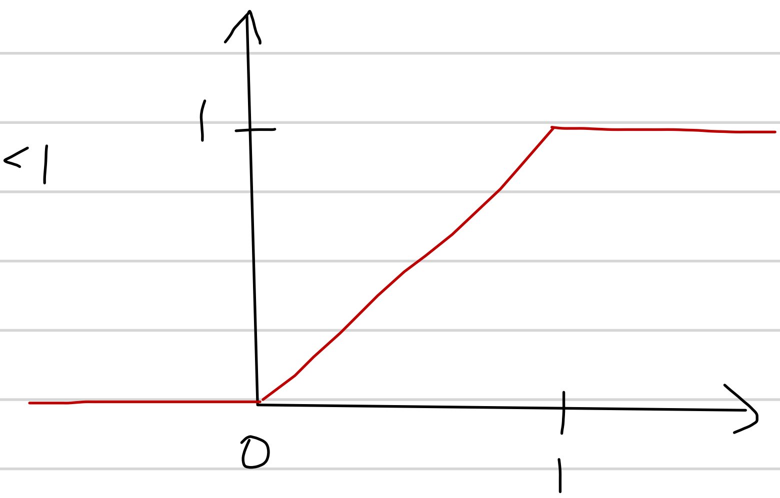
$\Rightarrow$



cdf

Ex) $X \sim \text{Uniform}[0,1]$

$$F_X(c) = \begin{cases} 0 & \text{if } c<0 \\ c & \text{if } 0 \le c < 1 \\ 1 & \text{if } c \ge 1 \end{cases}$$



- Using CDF, we can express the distribution of a r.v. that cannot be expressed with pmf.

* Properties of cdf

(1) Monotonicity: $F_X(c)$ is non-decreasing with c.

Why? If $a < b$, $\{X \le a\} \subset \{X \le c\} \Rightarrow F_X(a) \le F_X(b)$

(2) Limits at $\pm\infty$:

$$\lim_{c \to -\infty} F_X(u) = 0, \quad \lim_{c \to \infty} F_X(u) = 1$$

(3) Right Continuity: $\lim_{n \to \infty} F_X(c + \frac{1}{n}) = F_X(c)$

Why? $\{X \le c+1\} \supseteq \{X \le c+\frac{1}{2}\} \supseteq \{X \le c+\frac{1}{3}\} \supset \cdots \supset \{X \le c+\frac{1}{n}\}$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{X \le c + \tfrac{1}{n}\} = \{X = u\}$$

$$\Rightarrow \lim_{n \to \infty} F_X(c + \tfrac{1}{n}) = F_X(c).$$

Note: $\{X \le c-1\} \subset \{X \le c - \frac{1}{2}\} \subset \{X \le c - \frac{1}{3}\} \subset \{X \le c - \frac{1}{4}\} \subset \cdots$

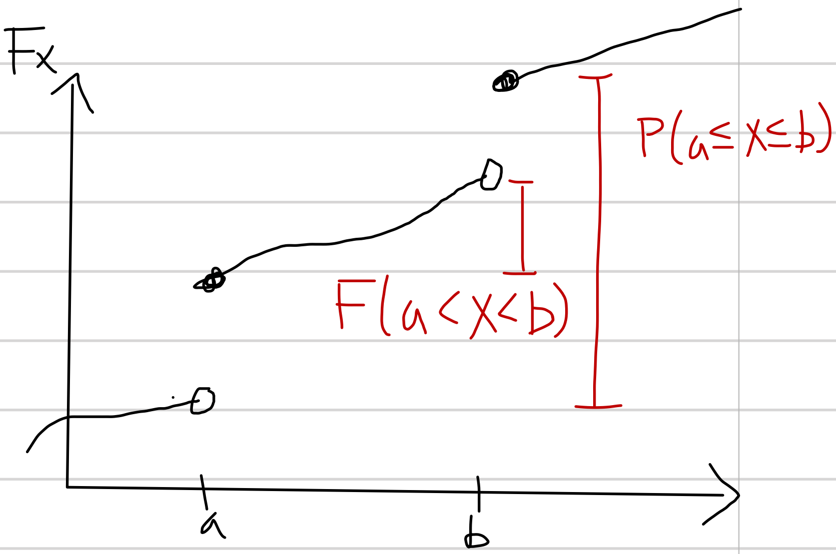
$$\Rightarrow \bigcup_{n=1}^{\infty} \{X \le c - \tfrac{1}{n}\} = \{X < c\}$$

$$\Rightarrow P(X<u) = \lim_{n\to\infty} F_X(c - \tfrac{1}{n})$$

$= F_X(c^-) \Rightarrow$ Left limit of F at c.

$$P(X=c) = P(X \le c) - P(X < c) = F_X(c) - F_X(c^-)$$

$$P(a \le X \le b) = F_X(b) - F_X(a^-)$$
$$P(a < X < b) = F_X(b^-) - F_X(a)$$



For a discrete-type R.V. X, the relationship between pdf and cdf:

$$F_X(c) = \sum_{u: u \le c} p_X(c)$$

Def) A r.v. X is called continuous-type if the CDF is an integral of a function $f_X(u)$, i.e.,

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\,du.$$

$f_X(u)$ is called the probability density function (pdf).