2.3 Conditional probability

Random experiment $(\Omega, \mathcal{F}, P) \Rightarrow P(A)$?

Additional Information about the outcome

Ex) Rolling a die : X = # of the die.

$A = \{\text{odd number comes out}\}$ $\quad P(A) = \frac{1}{2}$
$B = \{4 \text{ or } 5 \text{ or } 6 \text{ comes out}\}$. $\quad P(B) = \frac{1}{2}$

If you are not allowed to see the outcome, but are told that the outcome is one of $\{4,5,6\}$. What is the prob. that the outcome is odd?

$$\frac{1}{3} = \frac{P(\{5\})}{P(\{4,5,6\})}$$

If you are told that the outcome is even. What is the prob. that the outcome is 4 or higher?

$$= \frac{2}{3} = \frac{P(\{4,6\})}{P(\{2,4,6\})}$$

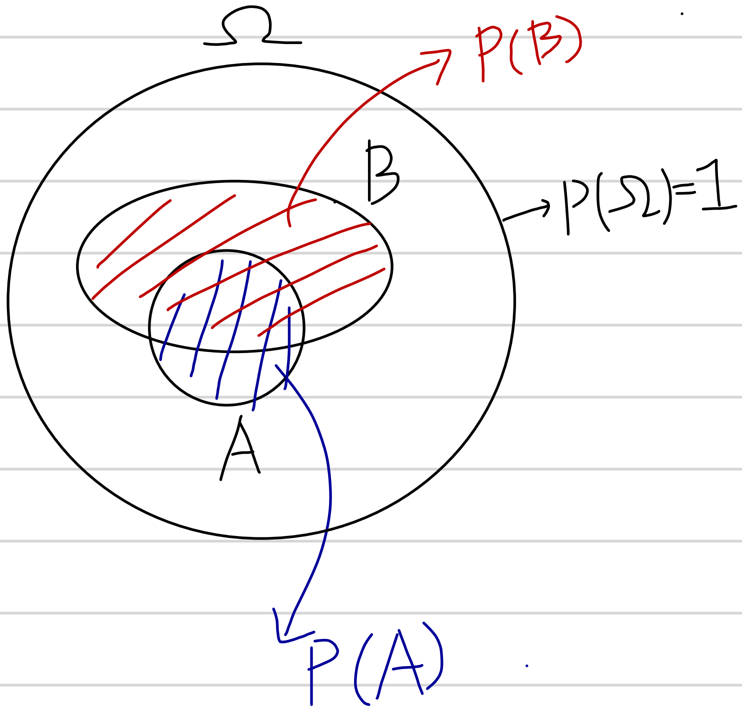
Def : Conditional probability of event A given by B

$$= P(A \mid B) = \begin{cases} \dfrac{P(AB)}{P(A)} & \text{if } P(A) > 0. \\ \text{undefined} & \text{if } P(B) = 0. \end{cases}$$

"The outcome of the die is higher than 7. What is the probability that the outcome is odd?"

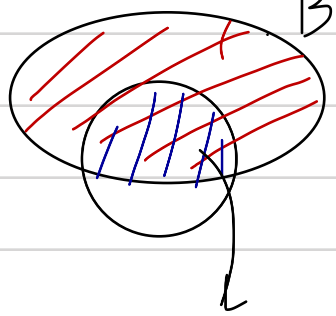
$\Rightarrow$ Liar! Not even possible!"

Interpretation



$$P(B|B) = \frac{P(B)}{P(B)} = 1$$

$B = \Omega'$



Measure of this area compared to P(B)? $\frac{P(AB)}{P(B)}$

Ex 2.3. 1) Rolling two dice

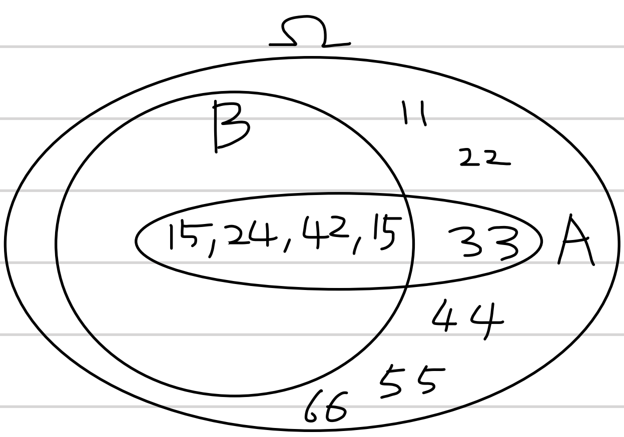
$A = \{\text{sum is six}\}$ $\quad$ $B = \{\text{numbers are not equal}\}$

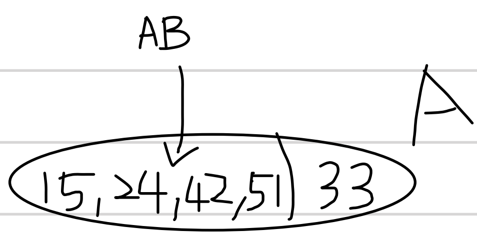
Compare $P(B)$ and $P(B|A)$

$$P(B) = 1 - P(B^c) = 1 - P(\text{numbers are the same}\}) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(A) = |\{15, 24, 33, 42, 15\}|/36 = \frac{5}{36}$$

$$P(AB) = |\{15, 24, 42, 15\}|/36 = 1/9$$





$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{4}{5}$$

Understand & Remember

Property of $P(\ |A)$ where $P(A) > 0$

1. $P(B|A) \geq 0$

2. $P(B|A) + P(B^c|A) = 1$

2.a For disjoint $E_1, E_2, \cdots, P(E_1 \cup E_2 \cup \cdots |A) = P(E_1|A) + P(E_2|A) + \cdots$

3. $P(\Omega|A) = \dfrac{P(\Omega \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

4. $P(AB) = P(A)P(B|A)$

5. $P(ABC) = P(C)\underbrace{P(AB|C)} = P(C)P(B|C)P(A|BC)$

$$\frac{P(ABC)}{P(C)} = \frac{1}{P(C)} \cdot P(A|BC) \cdot P(BC) = P(A|BC)\frac{P(BC)}{P(C)}$$

$$= P(A|BC)P(B|C)$$

EX) Poker: a deck C of 52 cards

$C = \{ i_j \mid i \in \{1, 2, \cdots, 13\}, j \in \{S, H, D, C\}\}$

$P(\text{TWO PAIR}) = 0.0475$

$\Rightarrow$ Suppose that you receives the cards with order.
(Does not change the probability)

$P(\text{Full House} \mid \text{first card is } 1C\}$?

$\Rightarrow$ A: The same

P(TWO PAIR | first two cards are 1C and 1S) ?

Have $\binom{50}{3}$ choices for the other three cards.

For the five cards to be TWO PAIR, the other three cards consists of one pair and one card with different numbers
Among these choices, # of ways to choose two same number ($\neq 1$) cards and one different number card?

- 12 ways to choose another number ($\neq 1$) for the two cards
- $\binom{4}{2}$ ways to choose two cards with the number
- 11 ways to choose the last number.
- 4 ways to choose one card with the number.

$$\Rightarrow \frac{12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{\binom{50}{3}} = \frac{12 \cdot \frac{4 \cdot 3}{2 \cdot 1} \cdot 44}{\frac{50 \cdot 49 \cdot 48}{3 \cdot 2 \cdot 1}} = 0.1616$$

2.4 Independence and the binomial distribution

"Independence?" of what? from what?

- Roll two distinct dice

  A: # shown in the first die is 4, 5, or 6.

  B: # shown in the second die is even

If you are informed whether or not A is true, will it help you to better predict or guess whether B is true?

No!

Prove it!

B

Ω =

11,	12,	13,	14,	15,	16
21,	22,	23,	24,	25,	26
31,	32,	33,	34,	35,	36
41,	42,	43,	44,	45,	46
51,	52,	53,	54,	55,	56
61,	62,	63,	64,	65,	66

A

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{9/36}{18/36} = \frac{1}{2} \qquad P(B) = \frac{1}{2}$$

Independence between events is sometimes unclear!

$C = \{\text{Sum is } 7\}$ $\Rightarrow$ C and A are independent?

$P(C) = 6/36 = 1/6$

$$P(C|A) = P(AC)/P(A) = \frac{3/36}{18/36} = \frac{3}{18} = \frac{1}{6}$$

$$\Rightarrow P(C|A) = P(C)$$

Def: $A, B \in \mathcal{F}$ are *mutually independent* if

$$P(AB) = P(A)P(B).$$

$$\cancel{\Downarrow} \quad \Uparrow$$

Why not $P(A|B) = P(A)$? Because $P(B)$ could be 0

$\Rightarrow P(A|B)$ is undefined

Independence of three or more events

Ex) Toss a coin twice

A: First outcome is H $\Rightarrow P(A) = \frac{1}{2}$

B: Second outcome is T $\Rightarrow P(B) = \frac{1}{2}$

C: Two outcomes are the same. $\Rightarrow P(AB) = \frac{1}{4}$

Pairwise independent

$$\frac{1}{4} = P(AB) = P(A)P(B)$$
$$= P(BC) = P(B)P(C)$$
$$= P(CA) = P(C)P(A)$$

Can we say that A, B, C are independent?

No!

AB and C independent?

→ If you know whether A and B are true, you can tell whether C is true!

Def: Events $A_1, A_2, \cdots, A_n$ are *independent* if

$$P(A_{i_1} \cdot A_{i_2} \cdot \cdots \cdot A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

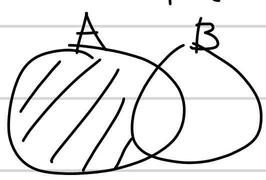
for any non-empty $\{i_1, i_2, \cdots, i_k\} \subset \{1, 2, 3, \cdots, N\}$

Claim: If A and B are independent, so are A and B^c?

<u>Proof</u>: Given $P(AB) = P(A)P(B)$,

$$P(AB^c) \stackrel{?}{=} P(A)P(B^c)$$

$$\begin{aligned} P(AB^c) &\\ \| \quad & \\ P(A) - P(AB) &= P(A) - P(A)P(B) \quad (\because A, B \text{ independent}) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$



A and B^c are independent!

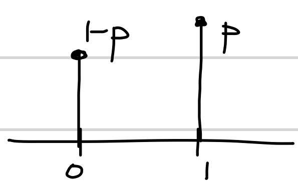
Q: If A, B, C are independent,

i) AB and C^c independent?
ii) $A \cup B$ and C^c independent?
iii) $A \setminus B$ and C^c independent? Exercise!

Bernoulli R.V.

A R.V. X with the following pmf is called Bernoulli RV with p (parameter)

$$p_X(i) = \begin{cases} p & \text{if } i = 1 \\ 1-p & \text{if } i = 0 \end{cases}$$

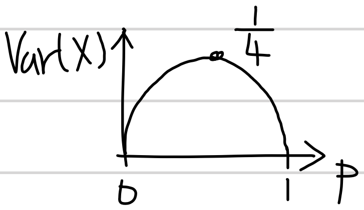


ex) Toss a coin. Let $X=1$ if Head comes out and $X=0$ otherwise.
$\Rightarrow X \sim \text{Bernoulli}(0.5)$
Roll a die. Let $X=1$ if six is shown in the die and $X=0$ otherwise
$\Rightarrow X \sim \text{Bernoulli}\left(\frac{1}{6}\right)$

Mean and Variance of a Bernoulli R.V. ?

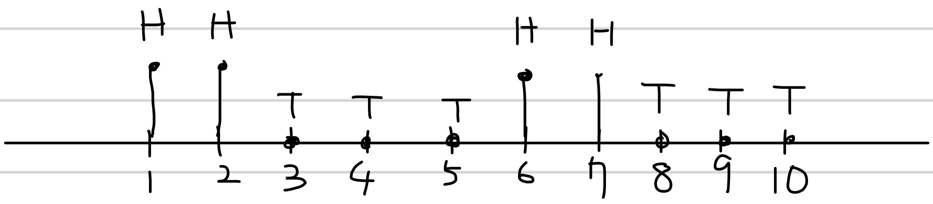
$$E[X] = \sum_{i \in \{0,1\}} i\, p_X(i) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$Var(X) = E[X^2] - E[X]^2 = 1^2 p + 0^2 (1-p) - p^2 = p - p^2 = p(1-p)$$



· Binomial R.V.

Toss a coin 10 times.



X = Sum of 10 independent Bernoulli (0.5)
= # of times that Head comes out.

$$p_X(k) = \binom{10}{k} 0.5^k \, 0.5^{10-k} = \binom{10}{k} 0.5^{10} \quad \text{for } k = 0, 1, \cdots, 10$$

# of ways to pick k times among $1, 2, \cdots, 10$ when the Heads comes out

Prob. that Heads come out at k specific times

Prob. that Tail comes out in the other times.

In general, a R.V. X is called Binomial R.V. with parameters n, p if the pmf of X is

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } 0 \le k \le n \\ 0 & \text{o/w} \end{cases}$$

- Binomial R.V. (n,p) ~ sum of n independent Bernoulli (p)'s.

- $$E[X] = \sum_{k=0}^{n} \cancel{k} \cdot \frac{n!}{(n-k)!\,\cancel{k!}} p^k (1-p)^{n-k}$$

$\Leftrightarrow k=1$ $\quad (k-1)!$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-k}$$

$k' = k-1$

$\Leftrightarrow k = k'+1$

$$= np \underbrace{\sum_{k'=0}^{n-1} \frac{(n-1)!}{(n-k'-1)!\,(k'-1)!} p^{k'} (1-p)^{n-1-k'}}_{\text{pmf of Binomial }(n-1,p)}$$

$$= np.$$

Var(X)?

$$E[X^2] = \sum_{k=0}^{n} k^{\cancel{2}} \frac{n!}{(n-k)!\,\cancel{k!}} p^k (1-p)^{n-k}$$

$\Leftrightarrow k=1$ $\quad (k-1)!$

$$= \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} k \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-k}$$

$k' = k-1$

$$= np \sum_{k'=0}^{n-1} (k'+1) \Bigg(\underbrace{\frac{(n-1)!}{(n-1-k')!\,k'!} p^{k'} (1-p)^{n-1-k'}}_{\text{Pmf of Binomial }(n-1,p)} \Bigg)$$

$$= np\, E[Y+1] \quad \text{where } Y \sim \text{Binomial }(n-1,p)$$

$$= np(E[Y]+1) = np((n-1)p+1) = n^2p^2 - np^2 + np$$

$$\therefore \mathrm{Var}(X) = E[X^2] - E[X]^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$$

Ex) Toss a coin 10 times, Prob that H comes out twice?

X: # of heads $\sim$ Binomial $(10, \frac{1}{2})$

$$p_X(2) = \binom{10}{2}\left(\frac{1}{2}\right)^{10} = \frac{10\cdot 9}{2\cdot 1} \times \frac{1}{1024} = \frac{45}{1024}$$

Ex) Best of Seven: Two teams A and B play a series of games

- In each game A wins w.p. $\frac{1}{2}$, (B loses)
  B wins w.p. $\frac{1}{2}$, (A loses)

- If a team wins 4 games first, the series ends and the team becomes the winner of the series.

Q1) How many games are possible? 7 games

Q2) Y = # of games before termination. $P_Y(k)$?

$P_Y(k) = 0$ if $k \le 3$. (At least four games are required!)

$P_Y(4) = P\{AAAA, BBBB\} = 2/16 = \frac{1}{8}$

$P_Y(5) = 2\,P(Y=5,\ A \text{ wins})$

$= 2\,P($ A wins three games at the first 4 games and
A wins the fifth game $\}$ indep.

$= 2\,P($A wins three games out of the first four games$\} \times$
$P($A wins the fifth game$\}$

$$= 2\cdot\binom{4}{3} 0.5^3 (1-0.5)^1 \cdot \frac{1}{2} = \cancel{2} \times \frac{4\cdot\cancel{3}\cdot\cancel{2}}{\cancel{3}\cdot\cancel{2}\cdot 1} \times 0.5^{\cancel{4}\,2} \cdot \frac{1}{\cancel{2}}$$

$$= 0.5^2 = \frac{1}{4}$$

$\Pr(6) = 2\,P(Y=6,\ A \text{ wins})$

$= 2\,P\{A \text{ wins three games at the first 5 games and } A \text{ wins the sixth game}\}$) indep.

$= 2\,P\{A \text{ wins three games out of the first five games}\} \times P\{A \text{ wins the sixth game}\}$

$= 2\cdot\binom{5}{3}\,0.5^3(1-0.5)^2\cdot\frac{1}{2} = \cancel{2}\cdot\frac{5\,\cancel{4}\,\cancel{3}}{\cancel{3}\,\cancel{2}\,1}\cdot\left(\frac{1}{2}\right)^{\cancel{6}\,4}$

$= \frac{5}{16}$

$\Pr(7) = 2\,P(Y=6,\ A \text{ wins})$

$= 2\,P\{A \text{ wins three games at the first 5 games and } A \text{ wins the seventh game}\}$) indep.

$= 2\,P\{A \text{ wins three games out of the first five games}\} \times P\{A \text{ wins the seventh game}\}$

$= 2\cdot\binom{6}{3}\,0.5^3(1-0.5)^3\cdot\frac{1}{2} = \cancel{2}\cdot\frac{\overset{2}{\cancel{6}}\,5\,\cancel{4}}{\cancel{3}\,\cancel{2}\,1}\cdot(0.5)^{\cancel{7}\,4}$

$= \frac{5}{16}$