Lecture 6

Countably infinite Vs. Uncountable.

Consider infinite sets

$$\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$$
$$\mathbb{Q}^+ = \{\tfrac{i}{j} : i, j \geq 1 \text{ are integers}\}$$
$$[0,1] = \{i : 0 \leq i \leq 1 \text{ is a real number}\}.$$

Do they have the same size ∞?

No! $|\text{Countably infinite set}| \ll |\text{uncountable set}|$

Def: Set S is *countably infinite* if all the elements is S can be placed in a list of sequence $\{a_1, a_2, \cdots\}$

$\mathbb{Z} \Rightarrow 0 \Rightarrow a_1,\ 1 \Rightarrow a_2,\ -1 \Rightarrow a_3,\ \cdots,\ k \Rightarrow a_{2k},\ -k \Rightarrow a_{2k+1}$

$\Rightarrow \mathbb{Z}$ is countably infinite

$\mathbb{Q}^+ = \{\frac{i}{j} \mid i, j \in \mathbb{Z}^+\}$ countable?

$$\begin{array}{cccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

The red line covers all $\frac{i}{j}$'s.

$\Rightarrow \mathbb{Q}^+$ is countably infinite.

Q: [0,1] countable?

Proof by contradition.

1. Assume that [0,1] is countable!

   ⇒ There exists list $a_1, a_2, \cdots$ that overall numbers in [0,1]

2. Show that there is a countradiction

$a_1 = 0.3439098\cdots$

$a_2 = 0.7106562\cdots$ $\qquad 0.4206\cdots?$

$a_3 = 0.5000000\cdots$

$a_4 = 0.2176523\cdots$

$a_{i,k}$: # number of a_i in the k-th decimal place.

$0.(a_{11}+1)(a_{22}+1)(a_{33}+1)\cdots$ where $(9+1)=0$

If this numbe is listed in a_i, the number in the ith decimal place ⇒ $(a_{ii}+1) \neq a_{ii}$ ⇒ Contradiction!

Recall that

$E[X+a] = E[X]+a$, $Var[X+a] = Var(X)$

$E[aX] = aE[X]$, $Var(aX) = a^2 Var(X)$.

Ex) Let $E[X] = \mu$ and $Var[X] = \sigma^2$.

$Y = 6 - 5X + X^2$

$$E[Y] = E[6-5X+X^2] = 6-5\mu+E[X^2] = 6-5\mu+Var(X)+E[X]^2$$
$$= 6-5\mu+\sigma^2+\mu^2$$

Ex) X takes 5 and 15 with prob. $\frac{1}{2}$ each.

$E[X] = 5\times\frac{1}{2} + 15\times\frac{1}{2} = 10$

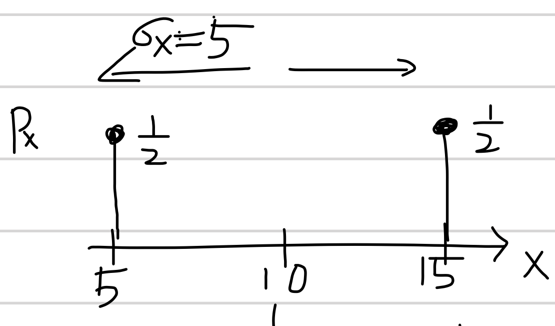
$Var[X] = E[X^2] - E[X]^2 = 5^2\frac{1}{2}+15^2\frac{1}{2}-10^2 \Rightarrow$?

$E[\underbrace{(X-E[X])}_{=g(X)}^2]$ is easier.

$\Rightarrow \pm 5$

$$= g^2(5)\frac{1}{2} + g^2(15)\frac{1}{2} = 25\frac{1}{2}+25\frac{1}{2} = \underline{\underline{25}}$$

$\sigma_X = 5$



↳ center of gravity

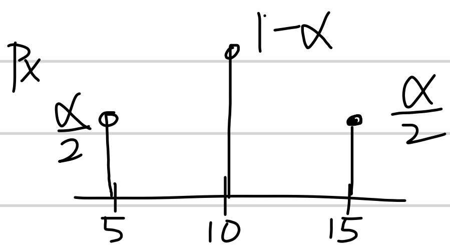
EX) X takes 5 w/p $\frac{\alpha}{2}$

$\quad$ 10 w/p $1-\alpha$

$\quad$ 15 w/p $\frac{\alpha}{2}$

α? $\quad \sum_u p_X(u) = 1 \Rightarrow 2\alpha = 1 \Rightarrow \alpha = \frac{1}{2}$

E[X] and Var[X]?



$\rightarrow$ Center of gravity = 10 = E[X].

Variance $= \underset{\sim}{5^2}$ No!

$$(X - E[X])^2 = \begin{cases} 5^2 & \text{with prob. } \alpha \\ 0 & \text{with prob.} \end{cases}$$

$\Rightarrow Var(X) = 5^2 \times \alpha \quad \Rightarrow \sigma_X = 5\sqrt{\alpha}$.