Lecture 05

Last time: Random Variable $X : \Omega \to \mathbb{R}$

- Discrete R.V.: $\exists\, u_1, u_2, \cdots, u_N$ or $u_1, u_2, \cdots$ s.t.
  $$P(X \in \{u_1, u_2, \cdots\}) = 1$$

- Probability Mass Function (pmf) $p_X(u) = P(X = u)$

- Mean of X: $E[X] = \sum_i u_i\, p_X(u_i)$
  where $u_1, u_2, \cdots$ are possible values of X

Q: $E[E[X]]$?

A: $E[E[X]] = E[X]$ Why? $E[X]$ is deterministic

Q: Interpretation of R.V. X?

$$\begin{tikzcd}
\text{Experiment} \arrow[rr, Rightarrow] \arrow[dr, bend right] & & \text{R.V. } X \\
& \text{outcome } \omega \in \Omega \arrow[ur, bend right] &
\end{tikzcd}$$

Perform the experiment n times.

$\Rightarrow X_1, X_2, \cdots, X_n$

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \approx E[X] \quad \text{with high probability}$$

ex) Rolling a die: $X = \#$ shown $\Rightarrow E[X] = 3.5$

Rolling n times: $Y =$ Average $\# = \dfrac{\text{sum of num.}}{n}$

$\Rightarrow Y \approx 3.5$ with high probability

[LOTUS] For a R.V. X and a real valued fct $g(\cdot)$,

$$E[g(X)] = \sum_i g(u_i) p_X(u_i)$$

ex) $E[X^2] = \sum_i u_i^2 p_X(u_i)$

[Linearity] $Z = X^2 + 4X$

$$E[Z] = \sum_u (u^2+4u) p_X(u) = \sum_u u^2 p_X + 4 \sum_u u\, p_X(u)$$

$$= E[X^2] + 4E[X]$$

- Short Questions: $E[g(X)] = g(E(X))$? No.

Common mistakes ex) $E[X^2] \neq E[X]^2$; $E\left[\frac{1}{X}\right] \neq \frac{1}{E[X]}$

Def: Variance of X : $Var(X) = E[(X-m)^2]$ where $m = E[X]$

Standard Deviation : $\sigma_X = \sqrt{Var(X)}$

Note (and remember)

$$Var(X) = E[(X-m)^2] = E[X^2 - 2mX + m^2] = E[X^2] - 2mE[X] + E[m^2]$$
$$= E[X^2] - 2m^2 + m^2 = E[X^2] - m^2 = E[X^2] - E[X]^2$$

Interpretation of σ_X ?

X is expected to be $E[X]$

σ_X : How much X deviates from $E[X]$.

} same unit

Ex) Select a student uniformly at random

X : height

What is $E[X]$? average height.

$x_1, x_2, \cdots, x_n$: height of students $1, 2, \cdots, N$

$$E[X] = \sum_{i=1}^{n} x_i P_X(x_i) = \sum_{i=1}^{n} x_i / n$$

note!

	$E[X]$	$Var(X)$	σ_X
In feet	5.7 feet	$0.09\,ft^2$	0.3 feet
In inches	58 inches	$16\,inch^2$	4 in

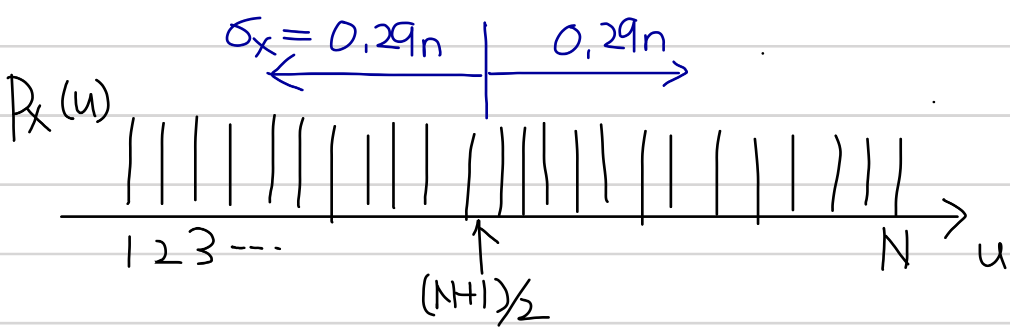
Ex) X takes a value from $1, 2, \cdots, n$ with equal prob.

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{n}\left(1^2+2^2+\cdots+n^2\right) - \left[\frac{1}{n}\left(1+2+\cdots+n\right)\right]^2$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right]$$

$$= (n+1)\left\{\frac{4n+2-3n-3}{12}\right\} = (n+1)(n-1)/12 = \frac{n^2-1}{12}$$

$$\sigma_X = \sqrt{\frac{n^2-1}{12}} \approx n\sqrt{\frac{1}{12}} \approx 0.29n$$

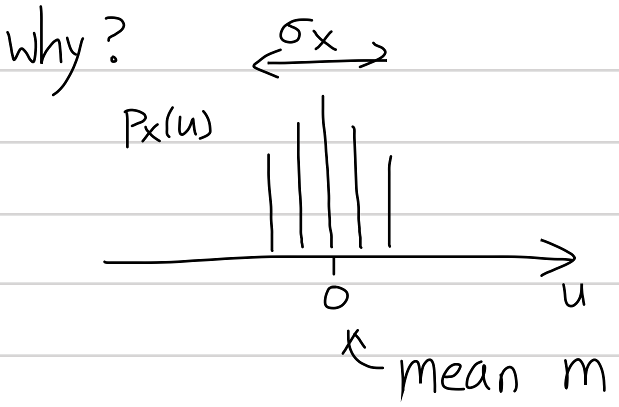


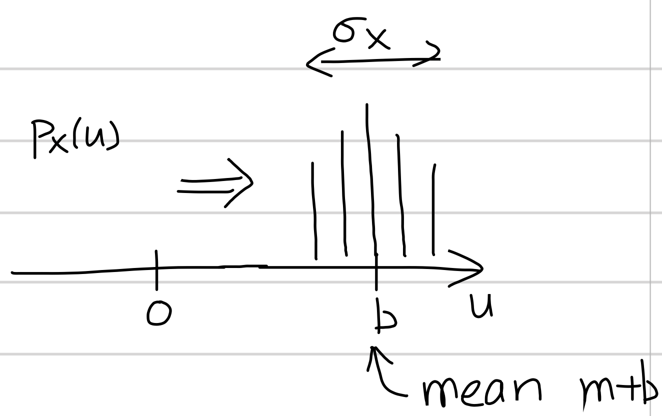
$$E[X+a] = E[X]+a$$

$$Var(X+a) = E[((X+b) - E(X+b))^2] = E[(X^2+b-E(X)-b)^2]$$

$$= E[(X-E[X])^2] = Var(X)$$

why?

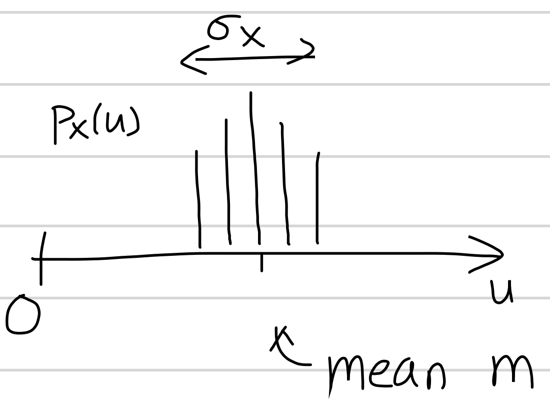




$$E[aX] = aE[X]$$

$$Var(X) = E[(aX - E(aX))^2] = E[(ax - aE[X])^2] = aE[(X-E[X])^2]$$

$$= a^2 Var(X)$$



expand a times!