Lecture 04

2. Discrete-type Random Variables

Today: Learn Random variables (RV)
Probability mass function (p.m.f)
Mean, Variance

What is a random variable?

Random Variable X: a real-valued function on Ω

$$X : \Omega \to \mathbb{R}$$

recall: Probability measure $P : \mathcal{F} \longrightarrow [0,1]$

- R.V. X is discrete-type if there exists a finite set $\{u_1, u_2, \cdots u_N\}$ or countably infinite set $\{u_1, u_2, \cdots\}$ such that

$$P(X \in \{u_1, u_2, \cdots\}) = 1 \quad \text{———} (*)$$

Probability mass function (pmf) p_X for a discrete R.V. X

$$p_X(u) = P\{X = u\} = P(\{\omega \in \Omega \mid X(\omega) = u\})$$

- From $(*)$, $\sum_i p_X(u_i) = 1$
- For any set A, $P(X \in A) = \sum_{u \in A} p_X(u)$

Ex 2.1.2) [Rolling a die]

X = Number of the die $\Rightarrow$ $p_X(i) = \frac{1}{6}$ for $1 \le i \le 6$.

Ex 2.1.4) [Rolling two fair dice]

$\Omega = \{(i,j) \mid 1 \le i \le 6,\ 1 \le j \le 6\}$

X : Sum of the two numbers, i.e., $X((i,j)) = i+j$

$\{X=4\} = \{(1,3), (2,2), (3,1)\}$

$\{X=5\} = \{(1,4), (2,3), (3,2), (4,1)\}$

$\vdots$

$$p_X(u) = \begin{cases} \frac{u-1}{36} & \text{for } 2 \le u \le 7 \\ \frac{13-u}{36} & \text{for } 7 < u < 12 \end{cases}$$

Y = Maximum of the two numbers showing, i.e.,

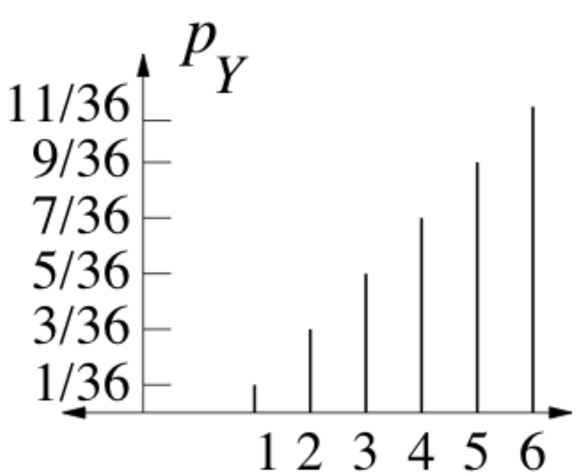
$Y((i,j)) = \max\{i, j\}$

pmf $p_Y(i)$?

$p_Y(3) = P(\{(1,3), (2,3), (3,3), (3,2), (3,1)\} = \frac{5}{36}$

$p_Y(i) = P(\{(1,i), (2,i), \cdots, (i,i), \cdots, (i,1)\}) = \frac{2i-1}{36}$

for $1 \le i \le 6$.



2.2 Mean and variance

Mean of R.V. X: $E[X] = \sum_i u_i \, p_X(u_i)$

where $u_1, u_2, \cdots$ are possible values of X

Ex 2.2.2) Rolling a die:

X: Shown number $\Rightarrow p_X(i) = \frac{1}{6}$ for $1 \le i \le 6$.

$$E[X] = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

Ex 2.2.3) Rolling three dice

$$\Omega = \{ i_1 i_2 i_3 \mid i_1, i_2, i_3 \in \{1, 2, \cdots, 6\} \}$$

Y: Number of distinct shown numbers

$\{Y=1\} = \{111, 222, 333, 444, 555, 666\}$

$\{Y=2\} = \{112, 121, 211, 113, 131, \cdots, 665, 656, 566\}$

$\{Y=3\} = \{ i_1 i_2 i_3 \in \Omega \mid i_1, i_2, i_3 \text{ are distinct} \}$

$$P_Y(1) = \frac{6}{6^3} = \frac{1}{36}$$

$$P_Y(3) = \frac{|\{Y=3\}|}{|\Omega|} = \frac{6 \times 5 \times 4}{6^3} = \frac{120}{6^3} = \frac{20}{36}$$

$$p_Y(2) = 1 - p_Y(1) - p_Y(2) = 1 - \frac{1}{36} - \frac{20}{36} = \frac{36 - 1 - 20}{36} = \frac{15}{36}$$

Ex 2.2.4) R.V. X is uniformly distributed in $\{-2,-1,0,\cdots,5\}$

$Y = X^2$. Find $E[Y]$

$\Rightarrow p_X(u) = \begin{cases} 1/8 & \text{if } u \in \{-2,-1,0,\cdots,5\} \\ 0 & \text{o/w} \end{cases}$

Two possible solutions:

1) $E[Y] = E[X^2] = \sum_{u=-2}^{5} u^2 p_X(u) = \dfrac{4+1+0+1+4+9+16+25}{8}$

$= \dfrac{60}{8} = 7.5$

2) Find $p_Y(u)$ $\Longrightarrow E[Y] = \sum_{u=0}^{25} u\, p_Y(u)$

u	$p_Y(u)$
0	1/8
1	2/8
4	2/8
9	1/8
16	1/8
25	1/8.

$= \dfrac{0\cdot1 + 1\cdot2 + 4\cdot2 + 9\cdot1 + 16\cdot1 + 25\cdot1}{8}$

$= \dfrac{60}{8} = 7.5$

Both ways result in the same answer.

∘ Mean of function $g(X)$ of R.V. X: [LOTUS]

$$E[g(X)] = \sum_i g(u_i)\, p_X(u_i)$$

where $u_1, u_2, \cdots$ are possible values of X.

[EXTENDED LOTUS]

For R.V.s $X_1, X_2, \cdots$ and functions $g_1, g_2, \cdots$.

$$E\left[\sum_i g_i(X_i)\right] = \sum_i E[g_i(X_i)]$$

Ex) Rolling two dice

$$\begin{cases} X_1 : \text{\# of the first outcome} \\ X_2 : \text{\# of the second outcome} \end{cases}$$

$X = X_1 + X_2$: sum of the numbers

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = \frac{21}{6} + \frac{21}{6} = 7$$