Lecture 2

Last time: Probability space $(\Omega, \mathcal{F}, P)$

- Ω: Set of all outcomes
- $\mathcal{F}$: Set of events (subset of Ω)
- P: Probability measure on events

Today: Important class of $(\Omega, \mathcal{F}, P)$

Ω : Finite, every outcome in Ω is equally likely.

$\Rightarrow$ for each $A \in \mathcal{F}$, $P(A) = \dfrac{|A|}{|\Omega|}$

Q: How to count the elements (outcomes) in A and Ω?

Ex) Simple example : Rolling a fair die :

Sample space : $\Omega = \{1,2,3,4,5,6\}$,

Event: 2 comes out $= \{2\} \Rightarrow P(\{2\}) = \frac{1}{6}$

An odd # comes out $= \{1,3,5\} \Rightarrow P(\{1,3,5\}) = \frac{1}{2}$

Principle of counting: # of ways to select $(w_1, w_2, \cdots, w_k)$

- n_1 ways to determine w_1
- n_2 " " w_2

$\vdots$

- n_k ways to determin w_k

$\Rightarrow$ Total $n_1 \times n_2 \times \cdots \times n_k$ ways

Ex 1.3.1) Random byte (8 bits : $x_1 x_2 x_3 x_4 \cdots x_8$)

$$2 \times 2 \times \cdots \times 2 = 2^8 = 256$$

Ex 1.3.2) # of orderings of A, B, C, D without repetition

eg) ABCD, DABC

In general,

$$\begin{matrix} X & X & X & X \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{matrix} \Rightarrow 4! \text{ (Factorial)}$$

Permutation: # of ways to order K elements from N distint elements without repetition :

$$\Rightarrow N \times (N-1) \times \cdots \times (N-K+1) = \frac{N!}{(N-K)!}$$

* Principle of over counting: If each element of a set is counted K times, the number of the elements of the set is the total count divided by K.

eg) How many orderings of letters P I Z Z A ?

$\Rightarrow 5!$? (**Incorrect**)

Ordering $P\,I\,Z_1 Z_2 A \Rightarrow 5!$ ways

we can switch Z_1 and Z_2 $\quad \begin{cases} P\,I\,Z_1 Z_2 A \\ P\,I\,Z_2 Z_1 A \end{cases}$

$$\Rightarrow \frac{5!}{2}$$

Ex 1.3.5) Nine players $A, B, \cdots, I$ in a basketball team

How many lineups of five players?

$(PG, SG, SF, PF, C) \Rightarrow 9 \times 8 \times 7 \times 6 \times 5$ (easy)

Lineup without position?

For given five players, $5!$ ways to assign positions

# of lineups without position? $\Rightarrow \dfrac{9\times 8\times 7\times 6\times 5}{5!}$

Combination: Among N distinct elements,

# of ways to choose K elements without repetition

$$\binom{N}{K} \triangleq \frac{N\times(N-1)\times\cdots\times(N-K+1)}{K!} = \frac{N!}{(N-K)!\,K!}$$

Note: $\binom{N}{K} = \binom{N}{N-K}$

Why? # of choosing K = # of not choosing $N-K$

Ex 1.4.3) POKER: For a given deck of 52 cards,

$C = \{1C, 2C, \cdots, 13C, 1D, 2D, \cdots, 13D, 1H, 2H, \cdots, 13H, 1S, 2S, \cdots, 13S\}$

five cards are given randomly. Pr(Full House)?

Sol) Sample space $\Omega = \{A \mid A \subset C \text{ and } |A| = 5\}$

$F = \{A \mid A \text{ is full house}\}$ (ex. $\{1S, 1D, 1H, 3H, 3S\}$)

$P(F) = \dfrac{|F|}{|\Omega|} = \dfrac{|F|}{\binom{52}{5}}$, $\qquad |F|$?

Sol)

- 13 ways to pick a number x_1 for the three cards
- 12 ways to pick a number x_2 for the two cards
- $\binom{4}{3}$ ways to pick three cards with x_1
- $\binom{4}{2}$ ways to pick two cards with x_2

$$\Rightarrow |F| = 13 \times 12 \times \binom{4}{3} \times \binom{4}{2}$$

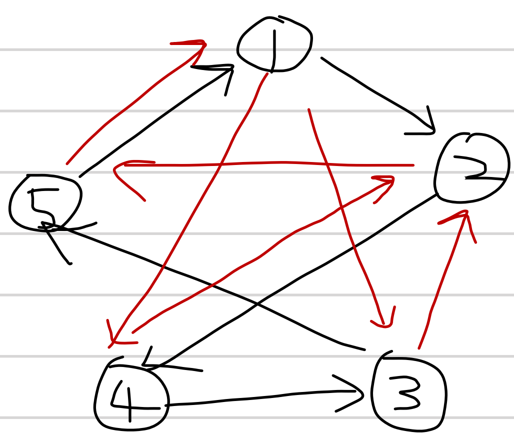
$$\Rightarrow P(F) = \frac{13 \times 12 \times \binom{4}{3} \times \binom{4}{2}}{\binom{52}{5}} = 0.0014$$

Q: P(TWO PAIR), P(STRAIGHT)?

Ex) Random Hamilton Dicycle: Suppose that there are nodes $1, 2, \dots, N$. We draw a Hamilton cycle by connecting all the nodes, such that each cycle is equally likely.

Pr(Nodes $1, 2, \cdots, K$ are not adjacent)?

Ex) $N=5$, $K=2$



Ω?

Let x_i be the node that is visited at the ith time.

$$\Omega \cancel{= \{} (x_1, x_2, \cdots, x_5) \mid x_i \in \{1, 2, \cdots, 5\},\ 1 \le i \le 5,\ x_i \ne x_j \text{ for } i \ne j \}$$

Correct? No! $(1,2,3,4,5), (2,3,4,5,1)$ are identical

For general N,

$$\Omega = \{ (1, x_2, \cdots, x_N) \mid x_i \in \{2, \cdots, 5\},\ 2 \le i \le n,\ x_i \ne x_j \text{ for } i \ne j \}$$

$$|\Omega| = (N-1)!$$

$$A = \{ (1, x_2, \cdots, x_N) \in \Omega \mid \text{nodes } 1, 2, \cdots, K \text{ are not adjacent} \}$$

$|A|$? $\Rightarrow$ How many ways to construct such a cycle?

1. Construct a cycle with node $K+1, K+2, \cdots, N$
2. Choose K edges where nodes $1, 2, \cdots, K$ to break in
3. Assign the chosen edges to the K nodes

$$|F| = \begin{cases} (N-K-1)! \binom{N-K}{K} K! = (N-K-1)! \dfrac{(N-K)!}{K!(N-2K)!} \cdot K! = \dfrac{(N-K-1)!(N-K)!}{(N-2K)!} & \text{for } K \le \frac{N}{2} \\ 0, & \text{for } K > \frac{N}{2}. \end{cases}$$

$$P(F) = \begin{cases} \dfrac{(N-K-1)!(N-K)!}{(N-2K)!(N-1)!} & \text{for } K \le \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$$