# Lecture 1

Objective : Learn how to characterize a random experiment using a probability space!

**Probability Space** : $(\Omega, \mathcal{F}, P)$

- $\Omega$ → all outcomes
- $\mathcal{F}$ → all events
- $P$ → Probability measure

**1. Sample space $\Omega$** : set of all possible outcomes

Eg) Coin tossing $\Omega = \{H, T\}$

Roll a die $\Omega = \{1, 2, 3, \cdots, 6\}$

Football (Purdue vs U of I) $\Omega = \{P, I, D\}$

**2. $\mathcal{F}$** : set of all subsets of $\Omega$ : set of all possible events

eg) Die : odd number $\{1, 3, 5\}$

Football : UoI does not lose $\{I, D\}$

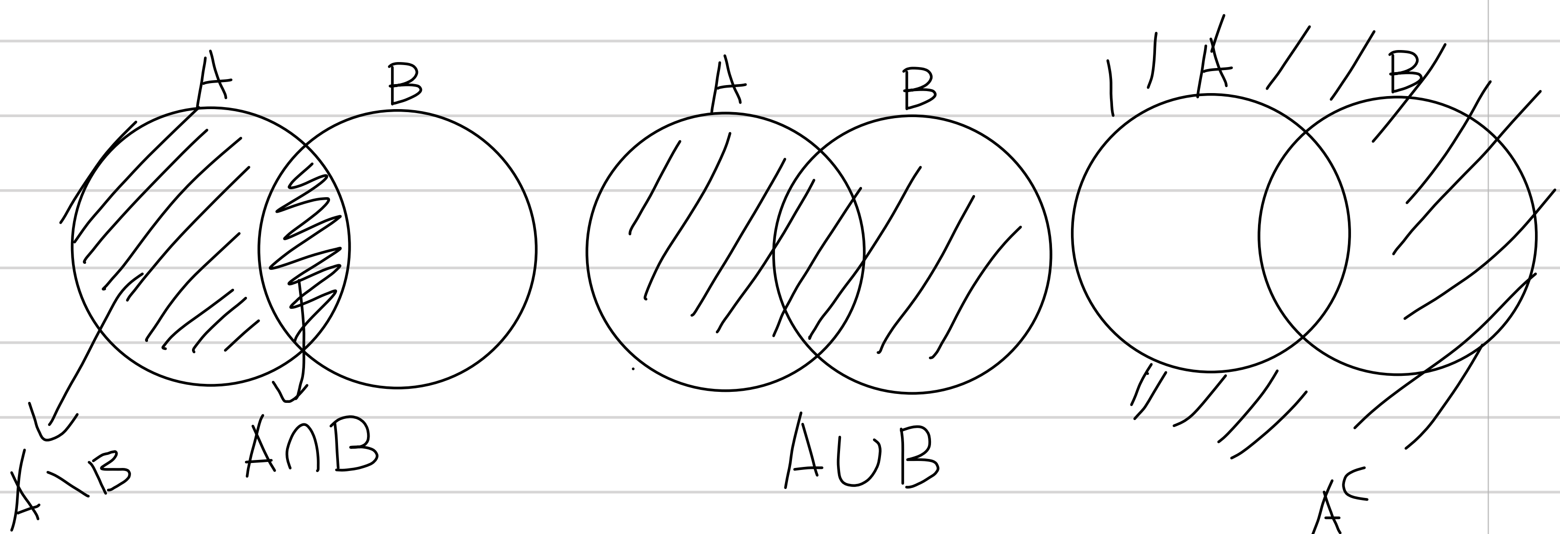
- Set operations

$AB = A \cap B \Leftrightarrow$ events $A$ and $B$ are true

$A \cup B \Leftrightarrow$ event $A$ or $B$ true

$A^c \Leftrightarrow A$ is not true

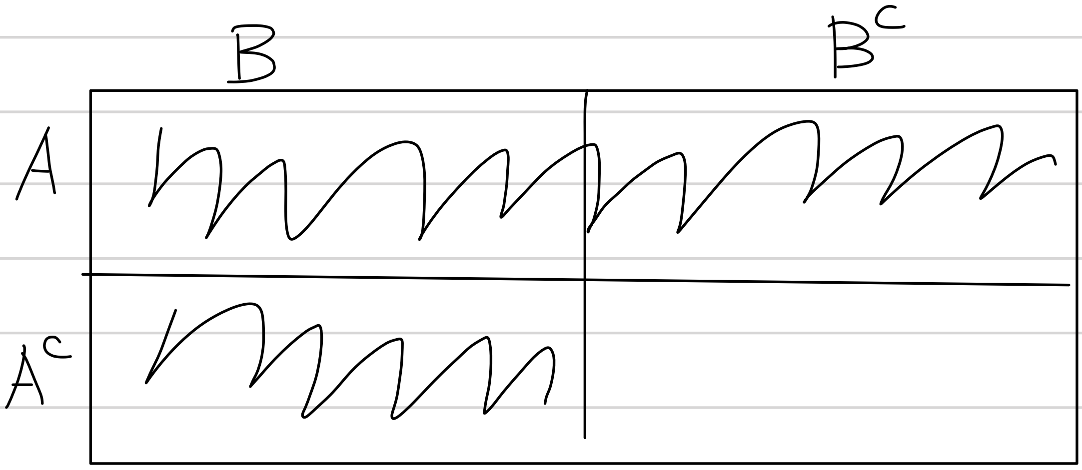
$A \setminus B = A \cap B^c \Leftrightarrow A$ is true, but $B$ is not!



# De Morgan's law

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

Karnaugh map !




$A \cup B$

Ex) Random Byte : bits in a byte are filled randomly

eg) 00001011, 10110011, . . .

$\Rightarrow$ Too many elements in $\Omega$

$\Rightarrow$ Conditional expression is needed!

$$\Omega = \{ \underbrace{(x_1, x_2, \cdots, x_8)}_{\text{element type}} \mid \underbrace{x_i \in \{0,1\},\ 1 \le i \le 8}_{\text{condition of the elements}} \}$$

Some events

equal bits = $\{00000000, 11111111\}$

sum is four = $\{(x_1, \cdots, x_8) \in \Omega \mid \sum_i x_i = 4\}$

Axioms: $\Omega \in \mathcal{F}$,

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (Hence, $\phi \in \mathcal{F}$)

$A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

How about $A \cup B$?

$A, B \in \mathcal{F} \Rightarrow A^c, B^c \in \mathcal{F} \Rightarrow A^c \cap B^c \in \mathcal{F}$

(De Morgan) $\Rightarrow (A \cup B)^c \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

- What is the probability of each event of $\mathcal{F}$?

**Probability Measure $P$:** $P : \mathcal{F} \longrightarrow [0,1]$

- Fair coin: $P(\{H\}) = P(\{T\}) = \frac{1}{2}$, $P(\{H, T\}) = 1$

To be a valid assignment, $P(\cdot)$ must satisfy the next axioms

Axiom 1: $P(A) \geq 0$ for every event $A \in \mathcal{F}$

Axiom 2: If $A \cap B = \phi$ (i.e., disjoint, mutually exclusive)

$$P(A \cup B) = P(A) + P(B)$$

Axiom 2(a): $E_1, E_2, \cdots$ are disjoint and countably infinite

$$\Rightarrow P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$$

Axiom 3: $P(\Omega) = 1$.

Consequences: $P(A^c) = 1 - P(A)$, $P(\phi) = P(\Omega^c) = 1 - P(\Omega) = 0$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex 1.2.3) Suppose you pick a real number on $[0,1]$

→ Probability space $(\Omega, \mathcal{F}, P)$ ?

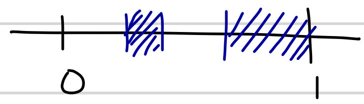
$\Omega = \{ \omega \mid 0 \le \omega \le 1 \}$

$\mathcal{F}$ : set of all open and closed coverings within $[0,1]$

"Not countable"

$P([a,b]) = b-a$.

$P($ Blue region $) =$




Ex 1.2.4) Repeated binary trials

Toss a coin infinitely

$w_i \in \{H, T\}$ : the outcome at the $i$-th trial

$\Omega = \{ (w_1, w_2, \cdots) \mid i \ge 1 \text{ and } w_i \in \{H,T\} \}$

→ Infinite # of outcomes!

Probability that first head occurs right after an even # of trials?

$E_k$ : Event that the first head comes out on the $k$th trial

$$P(F) = P(E_2 \cup E_4 \cup \cdots) = \sum_{k=1}^{\infty} P(E_{2k}) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k}$$

$$= \frac{1/4}{1-(1/4)} = \frac{1}{3}$$

Remember: $1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for $x \neq 1$

$\longrightarrow \frac{1}{1-x}$ as $n \to \infty$ for $|x| < 1$.