ECE 313
Midterm Exam III

July 25, 2012

Exam Time : 100 mins

Problem 1

(3 × 8 pts)
Consider random variables X and Y with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}, & 0 \le x \le 1,\ 0 \le y \le 1, \\ \frac{2}{3}, & 0 \le x \le 1,\ -1 \le y < 0, \\ 0, & \text{eles.} \end{cases}$$

Determine the following.

a) The marginal pdf of Y.

b) The conditional pdf $f_{X|Y}(x|y)$.

c) $P\left(X^2 + Y^2 \le 1\right)$.

Solution

a)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^1 \frac{1}{3}dx = \frac{1}{3}, & 0 \le y \le 1, \\ \int_0^1 \frac{2}{3}dx = \frac{2}{3}, & -1 \le y < 0, \\ 0, & \text{else.} \end{cases}$$

b) The conditional distribution is defined only for $-1 \le y \le 1$.
For $-1 \le y < 0$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2/3}{2/3} = 1, & 0 \le x \le 1, \\ \frac{0}{2/3} = 0, & \text{else.} \end{cases}$$

For $0 \le y \le 1$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1/3}{1/3} = 1, & 0 \le x \le 1, \\ \frac{0}{1/3} = 0, & \text{else.} \end{cases}$$

So, for $-1 \leq y \leq 1$,

$$f_{X|Y}(x|y) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{else.} \end{cases}$$

c)

$$P\left(X^2 + Y^2 \leq 1\right) = \frac{\pi}{4}\left(\frac{1}{3}\right) + \frac{\pi}{4}\left(\frac{2}{3}\right) = \frac{\pi}{4}.$$

Problem 2

(3 × 8 pts)

Suppose arrival of calls to a call center can be modeled with a Poisson process with arrival rate λ calls per minute.

a) Find the ML estimate of λ if it is observed that the second call arrives at time $t = 4$ min.

b) For this part and the following part of the problem assume $\lambda = 2$. What is the probability that in the first 2 minutes at least 3 calls arrive.

c) What is the conditional probability that two calls arrive in the first two minutes given that two calls arrive in the first minute?

Solution

a) The arrival time of the second call, T_2, has the following Erlang distribution

$$f_{T_2}(t) = \begin{cases} \lambda^2 t e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

The the likelihood is

$$f_{T_2}(4) = \lambda^2 4 e^{-4\lambda}.$$

To maximize, we take log, differentiate, and set equal to zero:

$$f_{T_2}(4) \xrightarrow{\ln} 2\ln\lambda + \ln 4 - 4\lambda \xrightarrow{\frac{d}{d\lambda}} \frac{2}{\lambda} - 4 \xrightarrow{=0} \frac{2}{\lambda} - 4 = 0 \Rightarrow \hat{\lambda}_{ML} = \frac{1}{2}.$$

b) The desired probability is $P(N_2 \geq 3)$.

$$\begin{aligned} P(N_2 \geq 3) &= 1 - P(N_2 = 0) - P(N_2 = 1) - P(N_2 = 1) \\ &= 1 - e^{-2\lambda} - e^{-2\lambda}(2\lambda) - e^{-2\lambda}\frac{(2\lambda)^2}{2} \\ &= 1 - e^{-4}(1 + 4 + 8) = 1 - 13e^{-4}. \end{aligned}$$

c) Let $X (= N_1)$ denote the number of calls arriving in the first minute and $Y = (N_2 - N_1)$ denote the number of calls arriving in the second minute.

$$\begin{aligned} P(X + Y = 2|X = 2) &= \frac{P(X + Y = 2, X = 2)}{P(X = 2)} = \frac{P(X = 2, Y = 0)}{P(X = 2)} \\ &= \frac{P(X = 2)P(Y = 0)}{P(X = 2)} = P(Y = 0) = e^{-2}, \end{aligned}$$

where the third equality follows from the independence of the number of arrivals in non-overlapping intervals.

Problem 3

(2 × 8 pts)
Let X be a uniform random variable over the interval $[-1, 1]$ and let $Y = e^{X^2}$.

a) Determine the support of Y.

b) Determine the pdf of Y over its support.

Solution

a) The support of Y is $\left[e^0, e^1\right] = [1, e]$.

b) Consider the equation $y_0 = g(x) = e^{x^2}$. The solutions are

$$\begin{cases} x_1 = \sqrt{\ln y_0} \\ x_2 = -\sqrt{\ln y_0} \end{cases}$$

So,

$$f_Y(y_0) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} = \frac{1/2}{\left|2x_1 e^{x_1^2}\right|} + \frac{1/2}{\left|2x_1 e^{x_2^2}\right|} = \frac{1}{2y_0\sqrt{\ln y_0}}.$$

Problem 4

(3 × 8 pts)
Consider the following binary hypothesis testing problem.
Under hypothesis H_1, X is a Normal random variable with mean 0 and variance 1, that is,

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Under hypothesis H_0, X is uniform over the interval $\left[-\sqrt{\frac{\pi e}{2}}, \sqrt{\frac{\pi e}{2}}\right]$, that is,

$$f_0(x) = \begin{cases} \frac{1}{\sqrt{2\pi e}}, & -\sqrt{\frac{\pi e}{2}} \leq x \leq \sqrt{\frac{\pi e}{2}} \\ 0, & \text{else.} \end{cases}$$

Determine the following. (Hint: sketch both pdfs. For sketching purpose only, you may use $\sqrt{\pi e/2} \simeq 2$, $\sqrt{2\pi e} \simeq 4$, $1/\sqrt{2\pi} \simeq 0.40$, $1/\sqrt{2\pi e} \simeq 0.25$; your solutions must be in terms of π and e.)

a) Find the maximum likelihood (ML) decision rule.

b) Find p_m for the ML decision rule in terms of the Q function with positive arguments.

c) Define $\pi_1 = P(H_1)$ and $\pi_0 = P(H_0)$. Find the minimum value of $\frac{\pi_1}{\pi_0}$ such that MAP always declares H_1 to be true.

Solution

a) The following figure is helpful for understanding the solution.

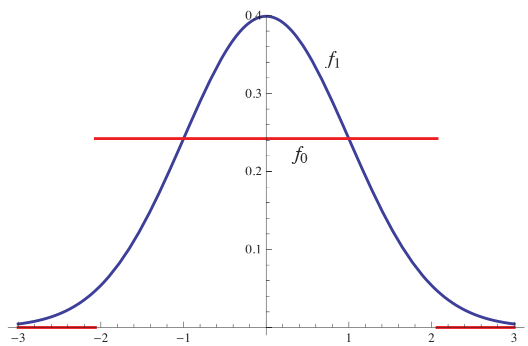

Figure 1: f_1 and f_0

Clearly, for $|X| > \sqrt{\frac{\pi e}{2}}$, ML declare H_1 to be true. Let γ be a non-negative value such that

$$f_1(\gamma) = f_0(\gamma).$$

Hence,

$$\frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} = \frac{1}{\sqrt{2\pi e}} \iff \gamma = 1.$$

So, the ML rule becomes

$$\begin{cases} X < -\sqrt{\frac{\pi e}{2}}, & \text{dec. } H_1 \\ -\sqrt{\frac{\pi e}{2}} < X < -1, & \text{dec. } H_0 \\ -1 < X < 1, & \text{dec. } H_1 \\ 1 < X < \sqrt{\frac{\pi e}{2}}, & \text{dec. } H_0 \\ \sqrt{\frac{\pi e}{2}} < X, & \text{dec. } H_1 \end{cases}$$

b) We have

$$p_m = P\left(1 < |X| < \sqrt{\frac{\pi e}{2}} | H_1\right) = 2Q(1) - 2Q\left(\sqrt{\frac{\pi e}{2}}\right).$$

c) π_1 and π_0 should be such that $\pi_1 f_1$ is always at least as large as $\pi_0 f_0$ as seen in the figure below. The minimum value of $\frac{\pi_1}{\pi_0}$ for which this condition is satisfied can be obtained as

$$\pi_0 \frac{1}{\sqrt{2\pi e}} = \pi_1 \frac{1}{\sqrt{2\pi}} e^{-\pi e/4} \iff \frac{\pi_1}{\pi_0} = e^{\pi e/4 - 1/2}.$$

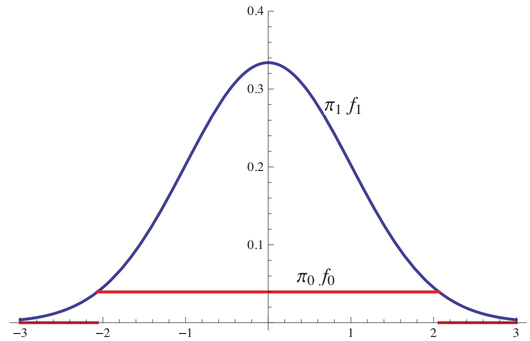

Figure 2: $\pi_1 f_1$ and $\pi_0 f_0$

Problem 5

(12 pts)

An airline sells 162 tickets for a plane with 120 seats. Each passenger actually shows up at the airport with probability 2/3. Using Gaussian approximation with continuity correction, what is the probability that there are not enough seats on the plane for all passengers who show up?

Solution

Let X denote the number of passenger who actually show up. We have

$$EX = 162\,(2/3) = 108,$$
$$\text{STD}\,(X) = \sqrt{162\,(2/3)\,(1/3)} = 6$$

Then, the desired probability is

$$P\,(X \geq 121) = P\,(X \geq 120.5) = P\left(\frac{X - 108}{6} \geq 2.08\right) = Q\,(2.08) = 0.0188.$$