University of Illinois | **Problem Set #8** | **ECE 313 Summer 2003**

Assigned: Thursday, July 31
Due: Wednesday, August 6
Reading: Ross Chapters 6.5, 7.1-7.5
Problems:

1. The number of hours **R** that a student spends **r**eading about probability in preparation for the ECE 313 Final Examination and the number of hours **S** that the student spends **s**leeping can be modeled as random variables with joint probability density function

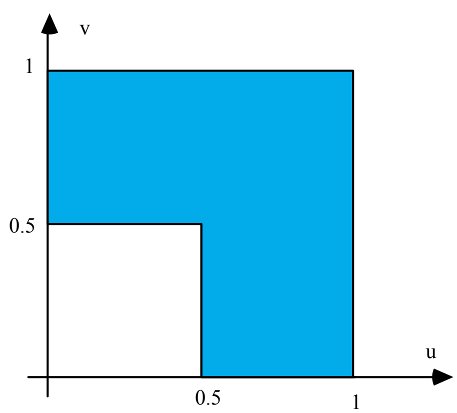
$$f_{\mathbf{R},\mathbf{S}}(x,y) = \begin{cases} K, 10 \le x + y \le 20,\ x \ge 0,\ y \ge 0, \\ 0, \text{otherwise.} \end{cases}$$

(a) What is the value of K?
(b) What is the marginal pdf of **R**?
(c) Unfortunately, the more the student tries to read about probability, the more confused the student gets. Also, the less the student sleeps, the more tired the student gets. As a result, the student's *percentage* score **T** on the Final Exam is related to **S** and **R** via the equation

$$\mathbf{T} = 50 + 2.5(\mathbf{S} - \mathbf{R}).$$

Find the pdf of **T**.

2. The random point (**X**,**Y**) is uniformly distributed on the shaded region shown below.
(a) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable **X**.
(b) *Write down* the marginal pdf $f_{\mathbf{Y}}(v)$ of the random variable **Y** from your answer to part (b).
(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.
(d) What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of **X** given that $\mathbf{Y} = \alpha$, if α satisfies $0 < \alpha < 1/2$?
What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of **X** given that $\mathbf{Y} = \alpha$, if α satisfies $1/2 < \alpha < 1$?
Now, apply the theorem of total probability to compute the unconditional pdf of **X** from $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$. Do you get the same answer as in part (a)?



3. Let the random variables **X** and **Y** be independent and uniformly distributed on (0,1). Find $E(|\mathbf{X}-\mathbf{Y}|)$ and $\mathrm{Var}(\mathbf{X}-\mathbf{Y})$.

4. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\mathrm{var}(\mathbf{X}) = 4$, $\mathrm{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$
(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?
(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\mathrm{cov}(\mathbf{T}, \mathbf{U})$?
(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\mathrm{var}(\mathbf{W})$.

(d) If **X** and **Y** are jointly Gaussian random variables, and **W** is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

5. This problem has three independent parts. Do not apply the numbers from one part to the others.

(a) If $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\mathrm{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\mathrm{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\mathrm{var}(\mathbf{X}) = 3\bullet\mathrm{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?

(b) If $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = \mathrm{var}(\mathbf{X} - \mathbf{Y})$, are **X** and **Y** uncorrelated ?

(c) If $\mathrm{var}(\mathbf{X}) = \mathrm{var}(\mathbf{Y})$, are **X** and **Y** uncorrelated ?

6. Consider the random point (**X**,**Y**) of Problem 2 above.

(a) Compute E[**X**] and var(**X**).

(b) Explain why the random variable **Y** has the same mean and variance as **X**.

(c) Compute E[**XY**] and hence find cov(**X**,**Y**).
should hold. ***Is*** the above equation satisfied by the numerical values you obtained?

(d) The conditional pdf of **X** given $\mathbf{Y} = \alpha$ was obtained in Problem 2 above, and it is easy to see that the conditional pdf of **Y** given $\mathbf{X} = \alpha$ is similar. Now, the **best** (least mean-square error) estimate of **Y** given $\mathbf{X} = \alpha$ is the mean of the conditional pdf of **Y** given $\mathbf{X} = \alpha$. Thus, if **X** has value $\alpha \le 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of **Y**, is 0.75 while if **X** has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of **Y** (given that **X** is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b were given in class. Compute a and b, and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α. (Remember that $0 \le \alpha \le 1$). For what value(s) of α are the two estimates the same?

(e) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of **X**, they really are *functions* of **X**, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, 0 \le \mathbf{X} \le 0.5, \\ 0.5, 0.5 < \mathbf{X} \le 1 \end{cases}$ and $\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?