Assigned: Thursday, July 24
Due: Thursday, July 31
Reading: Ross Chapters 6.1-6.4
Problems:

1. The probability of *heads* of a random coin is a random variable **P** uniform in the interval (0,1).

(a) Find $P\{0.3 \le \mathbf{P} \le 0.7\}$.

(b) The coin is tossed 10 times and *heads* shows 6 times. Find the posteriori probability that **P** is between 0.3 and 0.7.

2. We place at random 100 points in the interval (0,100). Find the probability that in the interval (0,0.1) there will be one and only one point

(a) Exactly

(b) Using the Poisson approximation

3. The discrete random variables **X** and **Y** have joint pmf $p_{\mathbf{X,Y}}(u,v)$ given by

4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
–1	1/12	1/6	1/12	0
v ↑ u→	0	1	3	5

(a) Find the marginal pmfs $p_{\mathbf{X}}(u)$ and $p_{\mathbf{Y}}(v)$ of **X** and **Y**.

(b) Are the random variables **X** and **Y** independent ?

(c) Find $P\{\mathbf{X} \le \mathbf{Y}\}$ and $P\{\mathbf{X} + \mathbf{Y} \le 8\}$.

(d) Find $p_{\mathbf{X}|\mathbf{Y}}(u|3)$, $E[\mathbf{X}|\mathbf{Y}=3]$, and $\text{var}(\mathbf{X}|\mathbf{Y}=3)$.

4. The number of α-particles emitted by a source during a unit time interval can be modeled as a Poisson random variable **X** with parameter λ. The α-particles are detected by means of a (imperfect) Geiger counter which detects a particle with probability $p < 1$. The detections of the various particles can be considered to be independent events. Thus, if n particles have been emitted, the Geiger counter reading can be modeled as a binomial random variable **Y** with parameters (n,p). In short, $p_{\mathbf{Y}|\mathbf{X}}(k|n)$, the *conditional* pmf of **Y** given that **X** = n, is a binomial pmf: $p_{\mathbf{Y}|\mathbf{X}}(k|n) = \binom{n}{k} p^k(1-p)^{n-k}$ for $0 \le k \le n$. (Note that $\mathbf{Y} \le \mathbf{X}$ always: the counter does not mistakenly count a particle when no particle is present, i.e. there are no false alarms! but the counter does occasionally fail to detect a particle).

(a) Sketch the u-v plane and the joint pmf of **X** and **Y**. Precision is not required in the sizes of the blobs you draw, but be sure that you don't put masses where they do not belong.

(b) What is the unconditional pmf of **Y**?

(c) What is the conditional pmf of **X** given **Y** = k?

5. The jointly continuous random variables **X** and **Y** have joint pdf given by

$$f_{\mathbf{X,Y}}(u,v) = \begin{cases} 2 \exp -(u+v), 0 < u < v < \infty, \\ 0, \text{ elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathbf{X,Y}}(u,v)$ is nonzero.

(b) Find the marginal pdfs of **X** and **Y**.

(c) Are the random variables **X** and **Y** independent ?

(d) Find $P\{\mathbf{Y} > 3\mathbf{X}\}$.

(e) For $\alpha > 0$, find $P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$.

(f) Use the result in part (e) to determine the pdf of the random variable $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

6. The jointly continuous random variables **X** and **Y** have joint pdf

$$f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} 1/2, 0 \le u < 1,\ 0 \le v < 1, \text{ and } 0 \le u + v < 1 \\ 3/2, 0 \le u < 1,\ 0 \le v < 1, \text{ and } 1 \le u + v < 2 \\ 0, \text{otherwise.} \end{cases}$$

Find $f_{\mathbf{X}}(u)$, $P\{\mathbf{X} + \mathbf{Y} \le 3/2\}$ and $P\{\mathbf{X}^2 + \mathbf{Y}^2 \ge 1\}$.

7. Let **X** and **Y** denote *independent* $\mathcal{N}(0, \sigma^2)$ variables.

(a) What is the joint pdf $f_{\mathbf{X},\mathbf{Y}}(u,v)$ of **X** and **Y**?

(b) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathbf{X}^2 + \mathbf{Y}^2 > \alpha^2\}$. Then, compute $P\{\mathbf{X}^2 + \mathbf{Y}^2 > \alpha^2\}$. Hint: read the Solutions to Problems 5(b) of Problem Set #1 and Problem 4(a) of Problem Set #6.

(c) Now, let $\mathbf{Z} = \mathbf{X}^2 + \mathbf{Y}^2$ denote the squared distance of the random point (**X**, **Y**) from the origin. Use the result of part (b) to deduce the pdf of **Z**.

From here onwards, assume $\sigma^2 = 1$ so that **X** and **Y** are independent *unit* Gaussian RVs.

(d) Express $P\{|\mathbf{X}| > \alpha\}$ in terms of the complementary unit Gaussian CDF function Q(x), and use this to write $P\{|\mathbf{X}| > \alpha, |\mathbf{Y}| > \alpha\}$ in terms of Q(x). (Remember commas mean intersections).

(e) Sketch the u-v plane and show on it the region over which you must integrate the joint pdf to find $P\{|\mathbf{X}| > \alpha, |\mathbf{Y}| > \alpha\}$. Compare the sketches in parts (b) and (d) to deduce that $P\{|\mathbf{X}| > \alpha, |\mathbf{Y}| > \alpha\} \le P\{\mathbf{X}^2 + \mathbf{Y}^2 > 2\alpha^2\}$.

(f) Show that the inequality of part (d) implies that $Q(x) \le (1/2)\bullet\exp(-x^2/2)$ as was proved earlier in Problem 4(b) of Problem Set #6.

8. Let (**X**, **Y**) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v)$ that is a circularly symmetric function, i.e., $f_{\mathbf{X},\mathbf{Y}}(u, v)$ can be expressed as g(r) where $r = \sqrt{u^2+v^2}$. The random point (**X**, **Y**) is at distance $\mathbf{R} = \sqrt{\mathbf{X}^2+\mathbf{Y}^2}$ from the origin.

(a) Show that for $\alpha \ge 0$, $P\{\mathbf{R} \le \alpha\} = \int_0^{\alpha} 2\pi\bullet r\bullet g(r)\, dr$.

(b) Use the formula for differentiating an integral that we studied in class to show that $f_{\mathbf{R}}(\alpha) = \frac{d}{du} P\{\mathbf{R} \le \alpha\} = 2\pi\bullet\alpha\bullet g(\alpha)$ for $\alpha > 0$.

(c) Use the result of part (b) to deduce the pdf of **R** if (**X**,**Y**) is uniformly distributed on the unit disc, viz. the interior of the circle of radius 1 centered at the origin

(d) Now suppose that (**X**, **Y**) has joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$

What is the value of C?

(e) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 0.25\}$.

9. Two resistors are connected in series to a one-volt voltage source. Suppose that the resistance values $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval (0, 1). Find the pdf $f_{\mathbf{I}}(a)$ of the current **I** (measured in amperes) in the circuit.

10. Let (**X**, **Y**) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

Find the pdf of $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$.