Assigned: Thursday, July 17
Due: Thursday, July 24
Reading: Ross Chapters 5.1-5.7, 9.1
Problems:

1. Let the random variable $\mathbf{I}_D$ denote the indicator function of an event D, that is,

$$\mathbf{I}_D(\omega) = \begin{cases} 1, \omega \in D, \\ 0, \omega \notin D. \end{cases}$$

Let A, B, and C denote independent events with probability 1/2, and define the random variable $\mathbf{X}$ by $\mathbf{X}(\omega) = \mathbf{I}_A(\omega) + 2\mathbf{I}_B(\omega) - \mathbf{I}_C(\omega)$.

(a) What are the values taken on by the random variable $\mathbf{X}$?

(b) Find the cumulative probability distribution function $F_{\mathbf{X}}(u)$ and the probability mass function $p_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$. Be very careful in specifying the values of $F_{\mathbf{X}}(u)$ at points where the function is discontinuous.

2. Consider a Poisson process with arrival rate μ. Let $\mathbf{X}$ denote the time of the first arrival after $t = 0$, and let τ denote a nonnegative real number.

(a) What is $P\{\mathbf{X} > \tau\}$?

(b) What is the probability density function of $\mathbf{X}$?

(c) Let A denote the event that there are exactly four arrivals in the time interval (0, 6]. What is P(A)?

(d) Let B denote the event that there are exactly two arrivals in the interval (4, 10]. What is P(AB)?

(e) What is the conditional probability that $\{\mathbf{X} > \tau\}$ *given* that the event A occurred? Be sure to give the answer for all nonnegative values of τ.

3. Raw scores on the SAT (and GRE) are transformed by a nonlinear function so that the minimum score is 200 and the maximum is 800. The histogram of scores *resembles* a Gaussian pdf with mean 500 and variance $ß^2 = 100^2$, that is, the score $\mathbf{X}$ of a student chosen at random can be modeled as a Gaussian random variable with mean 500 and variance $ß^2 = 100^2$. According to this model,

(a) What should your percentile rank be if your score is 700?

(b) What score corresponds to a percentile rank of 95% ?

(c) What fraction of students score between 300 and 550?

4. Let $\mathbf{X}$ denote a unit Gaussian random variable with pdf $\phi(u)$ and CDF $\Phi(u)$.

(a) What is the derivative of $\exp(-u^2/2)$ with respect to u? Use this result to find $E[|\mathbf{X}|]$.

Now, let $Q(x) = \int_x^\infty \phi(u)\, du = \int_x^\infty (\sqrt{2\pi})^{-1} \exp\left(-\frac{u^2}{2}\right) du = 1 - \Phi(x)$.

(b) A useful bound when x is small is $Q(x) \le (1/2)\exp(-x^2/2)$ for $x \ge 0$ in which equality holds only at $x = 0$. Derive this bound by first showing that $t^2 - x^2 > (t - x)^2$ for $t > x > 0$ and then applying this result to $\exp(x^2/2)Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1} \exp\left[-\left(\frac{t^2 - x^2}{2}\right)\right] dt$

5. Let **X** denote a unit Gaussian random variable.

(a) Find $E[|\mathbf{X}|]$.

(b) What is the conditional pdf of **X** given that $\mathbf{X} > 0$? i.e., what is $f_{\mathbf{X}|\mathbf{X}>0}(u|\mathbf{X} > 0)$?

(c) Suppose that **Y** is a random variable whose pdf just happens to be exactly the pdf you found in part (b). What is $E[\mathbf{Y}]$?

6. A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \leq 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X} - \mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X} - \alpha)^2 \text{if } \mathbf{X} > 0 \\ (\mathbf{X} + \alpha)^2 \text{if } \mathbf{X} \leq 0. \end{cases}$

So we want to choose α so that $E[\mathbf{Z}]$ is as small as possible. Use LOTUS to e-zily find $E[\mathbf{Z}]$ as a function of α, and then find the value of α that minimizes $E[\mathbf{Z}]$.

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, $\mathbf{W} = 3$ if $\mathbf{X} \geq 2.5$, $\mathbf{W} = 2$ if $1.5 \leq \mathbf{X} < 2.5$, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is $(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_0)$. What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

7. The lifetime of a VLSI chip can be modeled as an exponential random variable with parameter $\lambda = -\ln 0.999$/week.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the chip lasts for at least one week?
Now suppose that three identical chips are organized into a triple–modular–redundancy (TMR) system in which we assume that the majority–logic gate cannot fail. Furthermore, we assume that the three chips fail independently of one another, that is, if their lifetimes are $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, then the events $\{\mathbf{X}_1 > t_1\}$, $\{\mathbf{X}_2 > t_2\}$, and $\{\mathbf{X}_3 > t_3\}$ are independent for all t_1, t_2, t_3. Let **Y** denote the length of time for which the TMR system functions correctly.

(c) Express the event $\{\mathbf{Y} > t\}$ occurred in terms of unions, intesections and complements of the events $\{\mathbf{X}_1 > t\}$, $\{\mathbf{X}_2 > t\}$, and $\{\mathbf{X}_3 > t\}$.

(d) Show that $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. [Hint: $E[\mathbf{Y}]$ is the integral of $P\{\mathbf{Y} > t\}$ over the positive real line!] . Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

(e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?

(f) Find t such that $P\{\mathbf{Y} > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?

8. We continue from Problem 7, where you showed that *complementary* CDF of Y has the form: $1 - F_{\mathbf{Y}}(t) = P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$, $t > 0$.

(a) Sketch $f_{\mathbf{Y}}(t)$, the *pdf* of **Y** and also $f_{\mathbf{X}}(t)$, the pdf of **X**, the lifetime of a chip, on the same set of axes. Use Mathematica/Matlab etc. for this. Don't just wing it!

(b) Find all values of t for which $f_{\mathbf{Y}}(t) = f_{\mathbf{X}}(t)$. An analytical answer is desired. Don't just guesstimate the locations of the crossing-points of the two pdf curves from your graph in part (a). [Hint: set $x = \exp(-\lambda t)$]. For small values of t, which is larger: $f_{\mathbf{Y}}(t)$ or $f_{\mathbf{X}}(t)$?

(c) Sketch the hazard rate $h_{\mathbf{Y}}(t)$ of the TMR system and the hazard rate $h_{\mathbf{X}}(t)$ of a VLSI chip on the same set of axes. For what values of t does the TMR system have a larger hazard rate than the VLSI chip?

(d) Which is more likely to fail in the next minute: a one-week old TMR system or a one-week old VLSI chip?

(e) Which is more likely to fail in the next minute: a twenty-year old TMR system or a twenty-year old VLSI chip? For the purposes of this problem, twenty years = 1040 weeks.

9. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable **X** with pdf $f(u) = (bu)\bullet\exp(-bu^2/2)$ for $u > 0$. The complementary CDF is given by $P\{\mathbf{X} > t\} = \exp(-bt^2/2)$ for $t > 0$.

(a) Find the mean lifetime E[**X**] of the system using the formula $E[\mathbf{X}] = \int_0^\infty u\bullet f(u)du$.

(b) Use the result $E[\mathbf{X}] = \int_0^\infty P\{\mathbf{X} > t\}dt$ to find the mean lifetime of the system. Do you get the same answer as in part (a)? Why or why not?

(c) What is the median lifetime, and is it larger or smaller than the mean lifetime? How do these parameters compare to the mode of lifetime (mode = location of the pdf maximum)

(d) The system fails at time t, i.e. $\mathbf{X} = t$ is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?

10. **X** is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.

(a) What is the value of $P\{\mathbf{X} \le \ln 2\}$?

(b) Find the conditional probability that $P\{|\mathbf{X}| \le \ln 2\}$ given that $\{\mathbf{X} \le \ln 2\}$.

(c) Find the numerical value of $P\{\cos(\pi\mathbf{X}/2) < 0\}$.

(d) Now suppose that **X** denotes the voltage applied to a semiconductor diode, and that the current **Y** is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of **Y**.