Assigned: Friday, July 11
Due: Thursday, July 17
Reading: Ross Chapters 4.2, 4.9, 5.1-5.3

1. Which of the following are valid cumulative probability distribution functions (CDFs) ? For those that are not valid CDFs, state at least one property of CDFs which is not satisfied. For those which are valid CDFs, compute P{|**X**| > 0.5}.

(a) $F_{\mathbf{X}}(u) = \begin{cases} 0, u < 1, \\ 2u - u^2, 1 \le u \le 2, \\ 1, u > 2. \end{cases}$ **(b)** $F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), u < 0, \\ 1 - (1/4)\exp(-3u), u \ge 0. \end{cases}$

(c) $F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), u \le 0, \\ 1 - (1/4)\exp(-3u), u > 0. \end{cases}$

2. Let **X** denote the number of hours that a student works on ECE 340 each week. It is known that **X** is a *mixed* random variable with cumulative probability distribution function (CDF) $F_{\mathbf{X}}(u)$ given by

$$F_{\mathbf{X}}(u) = \begin{cases} 0, u < 0, \\ (1+u)/8, 0 \le u < 1, \\ 1/2, 1 \le u < 2, \\ 1/2 + u/8, 2 \le u < 4, \\ 1, u \ge 4. \end{cases}$$

Find the probability that the student

(a) works for exactly 2 hours,
(b) works for more than 2 hours,
(c) works for less than 2 hours,
(d) works for exactly 3 hours,
(e) works for more than 1/2 but less than 3 hours,
(f) works for more than 2 hours given that the student works at all, i.e. find P{**X** > 2|**X** > 0}.
(g) Find E[**X**].

3.(a) Prove that $E[\mathbf{X}] = \sum_{k=0}^{\infty} P\{\mathbf{X} > k\}$ for a discrete random variable **X** that takes on nonnegative integer values only.

(b) Find P{**X** > k} for k = 0, 1, 2, … for a geometric random variable **X** with parameter p. Substitute your answers in the formula of part (a) and give yet another derivation of the result that E[**X**] = 1/p.

4. Which of the following are valid probability density functions? Assume that the functions are zero outside the ranges specified. For those which are not valid pdfs, state at least one property of pdfs which is not satisfied. Also, state whether there exists a constant C such that Cf(u) is a valid pdf even though f(u) is not.

(a) f(u) = |u| for |u| < 1.
(b) f(u) = 1 – |u| for |u| < 1.
(c) f(u) = ln u for 0 < u < 1,
(d) f(u) = ln u for 0 < u < 2. Hint: ln u can be integrated by parts
(e) f(u) = 2u for 0 < u < 1.
(f) f(u) = (2/3)(u – 1) for 0 < u < 3.
(g) f(u) = exp(–2u), 0 < u < ∞,
(h) f(u) = 4 exp(–2u) – exp(–u), 0 < u < ∞.

5. The random variable **X** has probability density function

$$f_{\mathbf{X}}(u) = \begin{cases} \alpha(1-u), 0 < u < 1, \\ 0, \text{elsewhere}. \end{cases}$$

(a) Find $P\{6\mathbf{X}^2 > 5\mathbf{X} - 1\}$.

(b) Find $F_X(u)$. Be sure to specify the value of $F_X(u)$ for all u.

6. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable **X** with probability density function

$$f_X(u) = \begin{cases} 5(1-u)^4, & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

(a) If C = 0.5, (i.e., the tank holds 500 gallons) and **X** happens to have value 0.68 one particular week, (e.g. 680 people show up each wanting to purchase a gallon of gas for their snowblowers or lawnmowers), can the gas station satisfy the demand that week? That is, can the gas station supply gasoline to all those who want to buy it that week?

(b) If C = 0.5 and **X** happens to have value 0.43 some other week, can the gas station satisfy the demand during this other week? That is, can the gas station supply gasoline to all those who want to buy it that week?

(c) If C = 0.5, what is the *probability* that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666…, then, in the long run, the gas station can supply the weekly demand two weeks out of three.

(d) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

Suppose now that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let **Y** denote the amount of gasoline *sold* per week.

(e) How is **Y** related to **X**, the weekly *demand* for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!)

(f) What is the **average** weekly gross profit?

(g) Suppose that the owner pays \$20C as weekly rent on a tank of capacity 1000C gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the **average** weekly **net** profit and what value of C maximizes the average weekly net profit?

7. **X** is uniformly distributed on [–1, +1].

(a) If $\mathbf{Y} = \mathbf{X}^2$, what are the mean and variance of **Y**?

(b) If $\mathbf{Z} = g(\mathbf{X})$ where $g(u) = \begin{cases} u^2, & u \ge 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS (or the EZ method) to find E[**Z**]

(c) On a completely unrelated LOTUSian question, if **X** is a geometric random variable with parameter 1/2, and $\mathbf{Y} = \sin(\pi\mathbf{X}/2)$, what is the value of E[**Y**]?

8. The random variable **X** has probability density function $f_X(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1 - \mathbf{X})^2$.

(a) What is the minimum value of **Y**? Call this α.
What is the maximum value of **Y**? Call this β.
What do you think are the values of $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$?

(b) What is the CDF $F_Y(v)$ of the random variable **Y**?
Be sure to specify the value of $F_Y(v)$ for all v, $-\infty < v < \infty$.

(c) Show that the CDF $F_Y(v)$ that you found in part (b) is a nondecreasing continuous function. Is $F_Y(v)$ differentiable at α? at β?

(d) From the definition of the CDF $F_Y(v)$, we know that $P\{\mathbf{Y} \le \alpha - 1\} = F_Y(\alpha - 1)$ and

$P\{\mathbf{Y} > 2\beta\} = 1 - F_{\mathbf{Y}}(2\beta)$. Does substituting $v = \alpha - 1$ and $v = 2\beta$ in the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) give the same values for $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$ that you stated in part (a)? If the values are different, which ones are the correct values? Explain your choices of correct answer (and why the other possible answers are wrong) in detail.

9. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, 0 < \rho < 1, \\ 0 \text{elsewhere.} \end{cases}$

(a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius also have average volume? Does a sphere of average radius also have average surface area?

(b) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of **V**, the volume of the sphere.

(c) Find E[**V**] directly from this pdf. Do you get the same answer as in part (a)? Why not?

(d) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density **S** on the sphere ?

10. A newsboy purchases H newspapers for c_2 cents each and sells them for c_3 cents each. He can return unsold papers to the publisher for c_1 cents each. Note that $c_1 < c_2 < c_3$. The daily demand **X** for papers is a (nonnegative) integer-valued random variable with pmf $p_{\mathbf{X}}(u)$ and CDF $F_{\mathbf{X}}(u)$.

(a) What is the probability that he sells all H newspapers? Express your answer in terms of $p_{\mathbf{X}}(u)$ and also in terms of $F_{\mathbf{X}}(u)$.

(b) Let **Z** denote the daily profit (in cents) that the newsboy makes. Write an expression for **Z** in terms of **X** and H

(c) Use LOTUS to write an expression for his average daily profit. Your answer will depend on H, so call the expression for the **average** daily profit the function g(H).

(d) The newsboy has been buying H papers for some months and making an average profit g(H) each day. One day, he decides to buy one extra paper. What is the probability that he can sell this extra paper? Show that he makes an average **additional** profit of $(c_3 - c_2) - (c_3 - c_1)F_{\mathbf{X}}(H)$ from the extra paper. Call this A(H).

(e) Show that the average **additional** profit $A(H) = (c_3 - c_2) - (c_3 - c_1)F_{\mathbf{X}}(H)$ satisfies

$$\ldots A(H-1) \ge A(H) \ge A(H+1) \ldots,$$

that is, on average, each extra newspaper brings in smaller extra profit than the previous one. [Hint: $F_{\mathbf{X}}$ is a non-decreasing function…] This is called the law of diminishing returns.

(f) Show that for sufficiently large H, A(H) is negative so that the newsboy loses money (on the average) by buying too many extra papers.

(g) How many papers should he purchase to maximize his average profit?