1(a). **FALSE** Conditional probabilities can be smaller than or equal to the unconditional probabilities.
FALSE
TRUE Conditional probabilities satisfy axioms of probability.
TRUE The first term, P(B | A) P(A), is equal to P(A | B) P(B). This is just the theorem of total probability.
FALSE This does not hold unless P(A) = P(B) or P(AB) = 0.

(b) All four statements are true. $A\cap B\subset A \Rightarrow P(A\cap B) \le P(A)$, and $A\cap B\subset B \Rightarrow P(A\cap B) \le P(B)$. Thus, $P(A\cap B) \le \min\{P(A),P(B)\}$. Similarly, $A\subset A\cup B \Rightarrow P(A) \le P(A\cup B)$, and $B\subset A\cup B \Rightarrow P(B) \le P(A\cup B)$. Thus, $P(A)+P(B) \le 2P(A\cup B)$. Also, $1 \ge P(A\cup B) = P(A) + P(B) - P(A\cap B) \Rightarrow P(A\cap B) \ge P(A) + P(B) - 1$. Finally, $P(A \mid B) = P(A\cap B)/P(B)$, since $0 < P(B) < 1$, $P(A \mid B) \ge P(A\cap B)$.

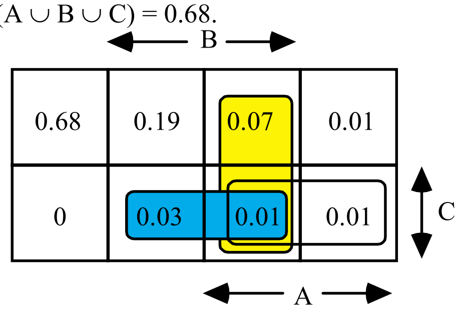
2. This problem is best solved with a Venn diagram or more preferably a Karnaugh map as shown below. From the given values of P(ABC) = 0.01, P(AB) = 0.08, P(BC) = 0.04, and P(AC) = 0.02, it is easy to find $P(ABC^c) = 0.08 - 0.01 = 0.07$, $P(A^cBC) = 0.04 - 0.01 = 0.03$, and $P(AB^cC) = 0.02 - 0.01 = 0.01$. Next, since $P(A) = 0.1 = P(ABC) + P(ABC^c) + P(AB^cC) + P(AB^cC^c)$, we get $P(AB^cC^c) = 0.01$. Similarly, we can show that $P(A^cBC^c) = 0.19$, and $P(A^cB^cC) = 0$. Finally, we can get
$P(A \cup B \cup C) = P(ABC) + P(ABC^c) + P(A^cBC) + P(AB^cC) + P(AB^cC^c) + P(A^cBC^c) + P(A^cBC^c)$ = 0.32. Crosscheck: Proposition 4.4 gives P(A) + P(B) + P(C) – P(AB) – P(AC) – P(BC) + P(ABC) which is also 0.32. Finally, $P(A^cB^cC^c) = 1 - P(A \cup B \cup C) = 0.68$. The probabilities asked for are

(a) P(only 1) = $P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC) = 0.01 + 0.19 + 0 = 0.2$,

(b) P(at least 2) = $P(AB \cup BC \cup AC) = P(ABC) + P(ABC^c) + P(A^cBC) + P(AB^cC)$ = 0.01 + 0.07 + 0.03 + 0.01 = 0.12. Crosscheck: $P(AB \cup BC \cup AC) = P(AB) + P(BC) + P(AC) - 2P(ABC)$

(c) P(B and at least one other) $P(B \cap (A \cup C)) = P(AB) + P(BC) - P(ABC) = 0.08 + 0.04 - 0.01 = 0.11$,

(d) P(no papers) $P(A^cB^cC^c) = 1 - P(A \cup B \cup C) = 0.68$.



3(a) **X** is a Binomial random variable with parameters (n=10,p=1/2). Its expected value, E[**X**], is np=5.

(b) For independent trials with a fair coin, probability of any particular toss being Head is just 1/2. So P(A)=1/2.

(c) If the event A has occurred, there has been a Head on the 4th toss. The number of Heads on the remaining 9 tosses other than the 4th toss is independent of A, and has average value 9×1/2=9/2. Hence, E[**X**|A]=1+9/2=11/2.

(d) B={no more than two Heads}={no Heads, or 1 Heads, or 2 Heads}. We first calculate P(B)=P(no Heads) + P(1 Heads) + P(2 Heads) = $\binom{10}{0}\frac{1}{2^{10}} + \binom{10}{1}\frac{1}{2^{10}} + \binom{10}{2}\frac{1}{2^{10}} = \frac{1+10+45}{2^{10}} = \frac{56}{2^{10}}$ .Next, we calculate P(AB)=P{4th toss is a Head and no more than two Heads}=P{4th toss is a Head and no Heads or 4th toss is a Head and 1 Head or 4th toss is a Head and 2 Heads}=P{4th toss is a Head and no Heads}+P{4th toss is a Head and 1 Head} + P{4th toss is a Head and 2 Heads} $= 0 + \frac{1}{2}\binom{9}{0}\frac{1}{2^9} + \frac{1}{2}\binom{9}{1}\frac{1}{2^9} = \frac{0+1+9}{2^{10}} = \frac{10}{2^{10}}$.

Hence, P(A| B) = 10/56=5/28.

4. By the theorem of total probability, P(**X**=2)=P(**X**=2|Both Fair)P(Both Fair) + P(**X**=2|1 Fair 1 Biased) P(1 Fair 1 Biased). P(Both Fair)=1/3, as there are a total of 3 ways of choosing 2 coins out of three, and only one way of choosing both coins fair. Similarly, P(1 Fair 1 Biased)=2/3. P(**X**=2|Both Fair)= $\binom{4}{2}\frac{1}{2^4} = \frac{3}{8}$,

since if both coins are fair, the probability we are looking for is that of observing 2 Heads on 2+2=4 tosses of a fair coin. To calculate P(**X**=2|1 Fair 1 Biased), note that a total of 2 Heads can occur in three different ways: The Fair coin lands Head twice, while the Biased one lands Tails twice, which has the probability

$\binom{2}{2}(\frac{1}{2})^2\binom{2}{0}(\frac{3}{4})^2 = \frac{9}{64}$. Or, the Fair coin may land Tails twice, while the Biased one lands Heads twice, which occurs with probability $\binom{2}{0}(\frac{1}{2})^2\binom{2}{2}(\frac{1}{4})^2 = \frac{1}{64}$. Or, the Fair coin and the Biased coin may land Heads exactly once each. There are $\binom{2}{1}\binom{2}{1} = 4$ ways this can happen resulting in a probability of $\binom{2}{1}\frac{1}{2}\frac{1}{2}\binom{2}{1}\frac{1}{4}\frac{3}{4} = \frac{12}{64}$. Thus, P(**X**=2|1 Fair 1 Biased) = $\frac{9}{64}+\frac{1}{64}+\frac{12}{64}=\frac{22}{64}$. Finally, plugging everything into the theorem of total probability yields P(**X**=2)= $\frac{3}{8}\frac{1}{3}+\frac{22}{64}\frac{2}{3}=\frac{17}{48}$.

5. Given that $\{\mathbf{X} \le 2\}$, either **X**=1 or **X**=2, since the geometric random variable takes on positive integer values. Let $\mathbf{Y}=(\mathbf{X}-1)^2$. Then, the condition $\{\mathbf{X} \le 2\}$ is equivalent to **Y**=0 or **Y**=1. If we let A denote the event that **Y**=0 or **Y**=1, the problem is asking E[**Y**|A]. To find this conditional expectation, we first find the conditional PMF of **Y** given A. Given A, **Y** takes on only two values: **Y**=0 and **Y**=1, with probailities
$\frac{P(Y=0)}{P(A)} = \frac{P(Y=0)}{P(Y=0)+P(Y=1)} = \frac{P(X=1)}{P(X=1)+P(X=2)} = \frac{5}{9}$,
$\frac{P(Y=1)}{P(A)} = \frac{P(Y=1)}{P(Y=0)+P(Y=1)} = \frac{P(X=2)}{P(X=1)+P(X=2)} = \frac{4}{9}$, respectively.Thus, E[**Y**|A]= $0\times\frac{5}{9}+1\times\frac{4}{9}=\frac{4}{9}$.