University of Illinois Spring 2020

ECE 313: Midterm Exam I

Wednesday, March 04, 2020
8:45 p.m. — 10:00 p.m.

1. **[20 points]** Suppose you and your friend play a simple gambling game in which you each roll a fair six-sided die. If you roll $n \in \{1,2,3,4,5,6\}$ then you win n dollars if your roll is higher than your friend's die. If it is lower, then you lose n dollars. If it is the same value, then there is no payout either way. Let X denote the money you win (which would be negative if you lose money).

   (a) **[8 points]** What is the expected payout $E[X]$ for you in this game?
   Solution:

$$
\begin{aligned}
E[X] &= \frac{1}{36}\sum_{n=1}^{6}\sum_{m<n} n - \frac{1}{36}\sum_{n=1}^{6}\sum_{m>n} n \\
&= -\frac{5}{36} - 2\times\frac{3}{36} - 3\times\frac{1}{36} + 4\times\frac{1}{36} + 5\times\frac{3}{36} + 6\times\frac{5}{36} \\
&= \frac{35}{36}.
\end{aligned} \tag{1}
$$

   (b) **[8 points]** What is $E[X^2]$?
   Solution: To compute the variance, we have

$$
E[X^2] = \sum_{n=1}^{6}\sum_{m\neq n} n^2\frac{1}{36} = \sum_{n=1}^{6} n^2\frac{5}{36} = 91\times\frac{5}{36} = \frac{455}{36}.
$$

   (c) **[4 points]** Express $\mathrm{Var}(aX+b)$ in terms of $E[X]$, $E[X^2]$, and constants a and b.
   Solution:

$$
\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) = a^2(E[X^2]-E[X]^2)
$$

2. **[20 points]**

   (a) **[6 points]** Suppose 6 letters {B,A,N,A,N,A} are placed in a hat and shuffled. You randomly choose one letter at a time without replacement. What is the probability that you will spell the word BANANA in the order you draw the letters? Assume that each letter has an equal probability of being drawn.
   Solution: In total there are 6! sequences that can be drawn with equal probability. If we assume the letters are labeled, then one such sequence will be $B_1A_2N_3A_4N_5A_6$. If we now remove the labels, this will allow for $3!\cdot 2!$ sequences that lead to the same spelling of BANANA, where we are accounting for a multiplicity of 3, and 2 on the letters A and N, respectively. Hence, the probability of choosing one sequence spelling BANANA is

$$
\frac{3!\cdot 2!}{6!} = \frac{1}{5\cdot 4\cdot 3} = \frac{1}{60}. \tag{2}
$$

(b) **[14 points]** Suppose again your goal is to spell BANANA by drawing the correct letters in the correct order, but now you draw with replacement on each letter until you get the right one. Once you get the correct letter, you do not replace it in the hat. What is the expected number of draws from the hat you must perform to correctly spell BANANA?

Note, you must spell it in the right order! For instance, you must first draw a B before you can move on to draw for an A, etc.

Solution: Let N_i be the number of draws taken to get the i^{th} letter in BANANA. Since we assume each letter has an equal probability of being drawn, the probability distribution $p_{N_i}(k)$ is a geometric distribution. Namely, $p_{N_i}(k) = (1-p_i)^{k-1}p_i$, where p_i is the probability of drawing the correct letter in the i^{th} slot. Since $E[N_i] = \frac{1}{p_i}$, we then compute

$$\begin{aligned} &E[N_1] = 6, && E[N_2] = \frac{5}{3}, && E[N_3] = \frac{4}{2} \\ &E[N_4] = \frac{3}{2}, && E[N_5] = \frac{2}{1}, && E[N_6] = \frac{1}{1} \end{aligned} \tag{3}$$

Therefore,

$$E[N_{tot}] = \sum_{i=1}^{6} E[N_i] = 6 + \frac{5}{3} + \frac{4}{2} + \frac{3}{2} + \frac{2}{1} + 1 = \frac{85}{6}$$

3. **[20 points]** There are two dice in a bag. One is a standard die with faces 1,2,3,4,5 and 6. The other one is a non-standard die with faces 2,2,4,6,6,6.

(a) **[10 points]** One die is selected at random from the bag and rolled. What is the probability that the number shown is less than or equal to 3?

Solution: Let E_1 be the event that the standard die is chosen and E_2 the event that the non-standard die is chosen. Let A be the event that we observe a number less than or equal to 3. By the law of total probability,

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{5}{12}.$$

(b) **[10 points]** The randomly selected die is rolled three times and the numbers shown are 2,2, and 4 (in this order). What is the conditional probability that the standard die was chosen?

Solution: Let B be the event that we observe 2,2, and 4. By Bayes' rule,

$$\begin{aligned} P(E_1|B) &= \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2)} \\ &= \frac{(1/6)^3(1/2)}{(1/6)^3(1/2) + (1/3)^2(1/6)(1/2)} = \frac{1/36}{1/36 + 1/9} = \frac{1}{5}. \end{aligned}$$

4. **[18 points]** You want to run a poll to estimate what fraction p of the population will vote for candidate A in the next elections. Suppose you interview n people, and out of those, X people say they will vote for candidate A.

(a) **[8 points]** What is the maximum likelihood estimate for p given $X = k$?

Solution: The number of people X that will vote for A can be modeled as a Binomial(n, p) random variable. As we saw in class, the ML estimate for p given X is $\hat{p}_{ML} = X/n$.

(b) **[10 points]** If you want to estimate p to within 0.01 with a confidence level of 96%, how large does n have to be? Please justify your answer.

Solution: We know that, for any $a > 0$,

$$P\left\{p \in \left(\hat{p}_{ML} - \frac{a}{2\sqrt{n}}, \hat{p}_{ML} + \frac{a}{2\sqrt{n}}\right)\right\} \geq 1 - \frac{1}{a^2}.$$

To obtain a 96% confidence, we choose $a = 5$. Then, in order to guarantee that $\hat{p}_{ML}$ is within 0.01 of p, we need $\frac{a}{2\sqrt{n}} \leq 0.01$, which implies $\sqrt{n} \geq 50 \cdot a = 250$, or $n \geq 62,500$.

5. **[22 points]** Suppose you keep flipping a coin until you observe a Head. The random variable X is the number of flips that is required. Based on the observation, you need to choose one of the following two hypothesis: H_0: it is a fair coin with $P(H) = 0.5$, and H_1: the coin is bent with $P(H) = \frac{2}{3}$.

(a) **[6 points]** Describe the ML decision rule. Express it in a simplified form.

Solution: The likelihood of observing $X = k$ under H_1 is $p_1(k) = \left(\frac{1}{3}\right)^{(k-1)} \frac{2}{3}$. The likelihood for $X = k$ under H_0 is $p_0(k) = \left(\frac{1}{2}\right)^k$. Therefore, the likelihood ratio is,

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{\left(\frac{1}{3}\right)^{(k-1)} \frac{2}{3}}{\left(\frac{1}{2}\right)^k} = 2 \times \left(\frac{2}{3}\right)^k \tag{4}$$

The ML decision rule is when $\Lambda(k) > 1$, we declare H_1 and if $\Lambda(k) < 1$, we declare H_0. Therefore, we have

$$\begin{cases} \text{Declare } H_1, & \text{if } k = 1, \\ \text{Declare } H_0, & \text{if } k \geq 2. \end{cases}$$

(b) **[6 points]** Describe the MAP decision rule under the assumption that H_1 is a priori twice as likely as H_0. Express it in a simplified form. (Hint: $\frac{\log(1/4)}{\log(2/3)} = 3.419$)

Solution: Since H_1 is twice as likely as H_0, the ratio of the prior probability $\frac{\pi_0}{\pi_1} = \frac{1}{2}$. According to the MAP decisoin rule, if $\Lambda(X) > \frac{1}{2}$, we declare H_1. In other words,

$$\begin{cases} \text{Declare } H_1, & \text{if } k \leq 3, \\ \text{Declare } H_0, & \text{if } k \geq 4. \end{cases}$$

(c) **[10 points]** Find the average error probability, p_e, for both the ML rule and the MAP rule, using the same prior distribution given in part (b). For which rule is the average error probability smaller?

Solution: Since $\pi_1 = 2\pi_0$ and $\pi_0 + \pi_1 = 1$, we get $\pi_1 = \frac{2}{3}$ and $\pi_0 = \frac{1}{3}$.
Under the ML decision rule, if $k = 1$, then we declare H_1.

$$p_{\text{false alarm}} = \sum_{k:\text{not declared as } H_0} p_0(k) = \frac{1}{2}$$

$$p_{\text{miss}} = \sum_{k:\text{not declared as } H_1} p_1(k) = \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^{(k-1)} \frac{2}{3} = 2 \times \frac{1}{9} \times \frac{1}{1 - \frac{1}{3}} = \frac{1}{3}$$

$$p_{e,\text{ML}} = \pi_0 \times p_{\text{false alarm}} + \pi_1 \times p_{\text{miss}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{3} = \frac{7}{18}$$

Under the MAP decision rule, if $k \leq 3$, then we declare H_1.

$$p_{\text{false alarm}} = \sum_{k:\text{not declared as } H_0} p_0(k) = \sum_{k=1}^{3} \left(\frac{1}{2}\right)^k = \frac{7}{8}$$

$$p_{\text{miss}} = \sum_{k:\text{not declared as } H_1} p_1(k) = \sum_{k=4}^{\infty} \left(\frac{1}{3}\right)^{(k-1)} \frac{2}{3} = 2 \times \left(\frac{1}{3}\right)^4 \times \frac{1}{1-\frac{1}{3}} = \frac{1}{27}$$

$$p_{e,\text{MAP}} = \pi_0 \times p_{\text{false alarm}} + \pi_1 \times p_{\text{miss}} = \frac{1}{3} \times \frac{7}{8} + \frac{2}{3} \times \frac{1}{27} = \frac{189+16}{81 \times 8} = \frac{205}{648}$$

MAP rule has a smaller average error probability, since $\frac{205}{648} < \frac{252}{648} = \frac{7}{18}$.